Assignment #11

Due on Wednesday, March 8, 2017

Read Section 4.2.2 on *Existence and Uniqueness* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Read Section 3.1 on *Properties of Linear Systems and the Linearity Principle* in
Blanchard, Devaney and Hall.

Read Section 3.6 on *Second–Order Linear Equations* in Blanchard, Devaney and
Hall.

Do the following problems

1. Prove that the initial value problem for the second order ODE

$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\[2mm] x(0) = x_o; \\ x'(0) = y_o, \end{cases}$$

   where b, c, x_o and y_o are given real constants, has a unique solution that exists
   for all $t \in \mathbb{R}$.

2. Let b and c be given real constants. Suppose that $x_1 \colon \mathbb{R} \to \mathbb{R}$ is a solution of
   the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\[2mm] x(0) = 1; \\ x'(0) = 0, \end{cases}$$

   and $x_2 \colon \mathbb{R} \to \mathbb{R}$ a solution of the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\[2mm] x(0) = 0; \\ x'(0) = 1. \end{cases}$$

   Prove that x_1 and x_2 are linearly independent.

   Suggestion: Begin with the equation

$$c_1 x_1(t) + c_2 x_2(t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

3. Let b and c be given real constants. Let $x_1 \colon \mathbb{R} \to \mathbb{R}$ and $x_2 \colon \mathbb{R} \to \mathbb{R}$ be linearly independent solutions of the second order differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0. \tag{1}$$

Prove that any solution of (1) must be of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \quad \text{for all } t \in \mathbb{R}.$$

4. Let b and c be given real constants. Suppose that λ_1 and λ_2 be distinct real solutions of the equation
$$\lambda^2 + b\lambda + c = 0.$$

Prove that the general solution of the equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

is given by
$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{for all } t \in \mathbb{R},$$

and arbitrary constants c_1 and c_2.

5. Find the solution of the initial value problem for the following second order differential equation:
$$\begin{cases} x'' - x = e^t; \\ x(0) = 1; \\ x'(0) = 0. \end{cases}$$