Assignment #10

Due on Wednesday, March 7, 2018

Read Section 4.2.2 on *Existence and Uniqueness* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 3.1 on *Properties of Linear Systems and the Linearity Principle* in Blanchard, Devaney and Hall.

Read Section 3.6 on *Second–Order Linear Equations* in Blanchard, Devaney and Hall.

Do the following problems

1. In this problem you will prove that the initial–value problem (IVP) for the second order ODE
$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\ \\ x(0) = x_o; \\ x'(0) = y_o, \end{cases} \tag{1}$$
where b, c, x_o and y_o are given real constants, has a unique solution that exists for all $t \in \mathbb{R}$.

   (a) Turn the system in (1) into an IVP for a two–dimensional system of first–order ODEs.

   (b) Apply the Existence and Uniqueness Theorem for Linear Systems proved in class and in the lecture notes to show that the IVP in part (a) has a unique solution that exists for all $t \in \mathbb{R}$.

   (c) Deduce that the IVP in (1) has a unique solution that exists for all $t \in \mathbb{R}$.

2. Let b and c be given real constants. Suppose that $x_1 \colon \mathbb{R} \to \mathbb{R}$ is a solution of the IVP
$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\ \\ x(0) = 1; \\ x'(0) = 0, \end{cases}$$

and $x_2\colon \mathbb{R} \to \mathbb{R}$ is a solution of the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0; \\ \\ x(0) = 0; \\ x'(0) = 1. \end{cases}$$

Prove that x_1 and x_2 are linearly independent.

Suggestion: Begin with the equation

$$c_1 x_1(t) + c_2 x_2(t) = 0, \qquad \text{for all } t \in \mathbb{R}.$$

3. Let b and c be given real constants. Let $x_1\colon \mathbb{R} \to \mathbb{R}$ and $x_2\colon \mathbb{R} \to \mathbb{R}$ be linearly independent solutions of the second order differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0. \tag{2}$$

Prove that any solution of (2) must be of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \qquad \text{for all } t \in \mathbb{R}.$$

4. Let b and c be given real constants. Suppose that λ_1 and λ_2 be distinct real solutions of the equation

$$\lambda^2 + b\lambda + c = 0.$$

Prove that the general solution of the equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

is given by

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad \text{for all } t \in \mathbb{R},$$

and arbitrary constants c_1 and c_2.

5. Find the solution of the initial value problem for the following second order differential equation:

$$\begin{cases} x'' - x = e^t; \\ x(0) = 1; \\ x'(0) = 0. \end{cases}$$