Assignment #15

Due on Wednesday, April 18, 2018

Read Section 6.3 on *Analysis of a Lotka–Volterra System* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.3 on *Hamiltonian Systems* in Blanchard, Devaney and Hall.

Do the following problems

1. Consider the system

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= x^3 - x. \end{cases} \tag{1}$$

   (a) Verify that the function $H\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}, \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{2}$$

   is a conserved quantity of the system in (1).

   (b) Sketch the level sets of the function H given in (2).

   (c) Sketch the phase portrait of the system in (1). Determine the nature of the stability of the equilibrium points.

2. For the system

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= x - x^2, \end{cases} \tag{3}$$

   (a) find a conserved quantity, H;

   (b) sketch the level sets of the function H found in part (a);

   (c) sketch the phase portrait of the system in (3), and determine the nature of the stability of the equilibrium points.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions with antiderivatives F and G, respectively.

   Five a conserved quantity for the system

$$\begin{cases} \dot{x} &= f(y); \\ \dot{y} &= g(x), \end{cases}$$

   in terms of the functions F and G.

4. For the Lotka–Volterra system

$$\begin{cases} \dot{x} & = & x(1-y); \\ \dot{y} & = & y(x-1), \end{cases}$$

let $(x(t), y(t))$ be a parametrization of a closed orbit in the first quadrant with period T. Verify that

$$\frac{1}{T}\int_0^T x(t)\ dt = \frac{1}{T}\int_0^T y(t)\ dt = 1.$$

Generalize this result for the case of the system

$$\begin{cases} \dot{x} & = & \alpha x - \beta xy; \\ \dot{y} & = & \delta xy - \gamma y, \end{cases} \tag{4}$$

where α, β, γ and δ are positive parameters.

5. Let α, β, γ and δ be the positive parameters in system (4) in Problem 4, and denote by Q_1^+ the set

$$Q_1^+ = \{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$$

Define $H\colon Q_1^+ \to \mathbb{R}$ to be

$$H(x,y) = \delta x - \gamma\ln(x) + \beta y - \alpha\ln(y), \qquad \text{for } (x,y) \in Q_1^+. \tag{5}$$

(a) Verify the function H given (5) is a conserved quantity for the system (4) in Problem 4.

(b) Show that H in (5) has a unique critical point in Q_1^+ and show that it H attains its minimum in Q_1^+ at that point.

(c) Sketch the level sets of the function H given in (5).

(d) Sketch the phase portrait of the system in (4). Determine the nature of the stability of the equilibrium points.