Assignment #5

Due on Wednesday, February 7, 2018

Read Section 4.1.3 on *Non–Diagonalizable Systems* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.1.4 on *Non–Diagonalizable Systems with One Eigenvalue* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.3 on *Phase Portraits of Linear Systems with Real Eigenvalues* in Blanchard, Devaney and Hall.

Do the following problems

1. Give the equations for the solution curves and sketch the phase portrait of the system $\left\{\begin{array}{rcl} \dot{x} & = & -y; \\ \dot{y} & = & x - 2y. \end{array}\right.$

2. Let $a\colon J \to \mathbb{R}$ denote a continuous function defined in a open interval J. Use separation of variables to find a formula for the solutions of the first order differential equation $\dfrac{dy}{dt} = a(t)y, \quad$ for $t \in J$.

3. Let a be as in Problem 2 and let $b\colon J \to \mathbb{R}$ be also a continuous function defined in J. Consider the first order differential equation

$$\frac{dy}{dt} = a(t)y + b(t), \qquad \text{for } t \in J. \tag{1}$$

   (a) Explain why, in general, the method of separation of variables does not apply to the differential equation in (1). State conditions under which separation of variables can be applied to solve the equation.

   (b) Let t_o be a point in J and define $A(t) = \displaystyle\int_{t_o}^{t} a(\tau)\, d\tau$ for all $t \in J$; that is, A is an antiderivative of a. Put $\mu(t) = e^{-A(t)}$, for all $t \in J$, and verify that the equation (1) can be written as

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)b(t), \qquad \text{for } t \in J. \tag{2}$$

   (c) The equation in (2) can be integrated and solved for μy. Use this procedure to obtain a formula for solutions of the equation in (1).

4. Use the procedure outlined in Problem 3 to obtain solutions to the first–order differential equation

$$\frac{dy}{dt} = -\frac{1}{t}y + 1, \qquad \text{for } t > 0.$$

Take $t_o = 1$.

5. In this problem we come up with solutions of the system

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} & = & \alpha x - \beta y; \\ \\ \dfrac{dy}{dt} & = & \beta x + \alpha y, \end{array}\right. \tag{3}$$

where $\alpha^2 + \beta^2 \neq 0$ and $\beta \neq 0$.

(a) Make the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and verify that

$$\begin{array}{rcl} \dot{r} & = & \dot{x}\cos\theta + \dot{y}\sin\theta, \\ \dot{\theta} & = & \dfrac{\dot{y}}{r}\cos\theta - \dfrac{\dot{x}}{r}\sin\theta. \end{array} \tag{4}$$

(b) Use the result in (4) to transform the system (3) into a system involving r and θ. Solve the system for r and θ.

(c) Based on your solution to part (b), give the general solution of the system (3).

(d) Sketch the phase portrait of the system in (3) for $\alpha = 0$ and $\beta = 1$.