## Assignment #6

**Due on Friday, February 23, 2018**

**Read** Section 4.1.5 on *Non–Diagonalizable Systems with No Real Eigenvalues* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Read** Section 3.4 on *Complex Eigenvalues* in Blanchard, Devaney and Hall.

**Background and Definitions.**
**Some Facts from the Arithmetic of Complex Numbers.**
The object $z = a + bi$, where $a$ and $b$ are real numbers, denotes a complex number; we write $z \in \mathbb{C}$. The complex number $i$ has the property that $i^2 = -1$.

- **Real Part of a Complex Number.** The real part of $z = a + bi$, denoted by $\text{Re}(z)$, is defined by
$$\text{Re}(z) = a.$$

- **Imaginary Part of a Complex Number.** The imaginary part of $z = a + bi$, denoted by $\text{Im}(z)$, is defined by
$$\text{Im}(z) = b.$$

- **Conjugate of a Complex Number.** The conjugate of $z = a + bi$, denoted by $\overline{z}$, is defined by
$$\overline{z} = a - bi.$$

  **Important Observations:**

  $z \in \mathbb{C}$ is a real number if and only if $\overline{z} = z$.
  $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$.
  For any complex numbers $z$ and $w$, $\overline{zw} = \overline{z}\,\overline{w}$.

- **Modulus of a Complex Number.** The modulus of $z = a + bi$, denoted by $|z|$, is defined by
$$|z| = \sqrt{a^2 + b^2}.$$
  Note that $z\overline{z} = |z|^2$.

- **Argument of a Complex Number.** The argument of $z = a + bi$, denoted by $\arg(z)$, is defined by
$$\arg(z) = \arctan\left(\frac{b}{a}\right).$$

**Do** the following problems

1. Let $A$ denote a $2\times 2$ matrix with real entries. Assume that $\lambda\in\mathbb{C}$ is a complex eigenvalue of $A$ with (complex) eigenvector $w$; that is, $w=\begin{pmatrix} z_1\\ z_2\end{pmatrix}$, where $z_1,z_2\in\mathbb{C}$. Show that $\overline{\lambda}$ is also an eigenvalue of $A$ with corresponding eigenvector $\overline{w}=\begin{pmatrix}\overline{z_1}\\ \overline{z_2}\end{pmatrix}$.

2. Let $A$ denote a $2\times 2$ matrix with real entries. Assume that $\lambda=\alpha+i\beta$ is a complex eigenvalue of $A$, where $\beta\neq 0$. Show that the characteristic polynomial of $A$ is given by $p_{_A}(\lambda)=\lambda^2-2\alpha\lambda+\alpha^2+\beta^2$.

3. Let $A$ denote a $2\times 2$ matrix with real entries with eigenvalues $\lambda_1=\alpha+i\beta$ and $\lambda_2=\alpha-i\beta$, with $\beta\neq 0$. Let $w_1\in\mathbb{C}^2$ be an eigenvector corresponding to $\lambda_1$.

   (a) Show that $w_2=\overline{w_1}$ is an eigenvector corresponding to $\lambda_2$.

   (b) Define vectors $v_1,v_2\in\mathbb{R}^2$ by
   $$v_1=\mathrm{Im}(w_1)=\frac{1}{2i}(w_1-w_2)\quad\text{ and } v_2=\mathrm{Re}(w_1)=\frac{1}{2}(w_1+w_2).$$
   Show that the set $\{v_1,v_2\}$ is linearly independent.

   (c) Verify that
   $$\begin{aligned} Av_1 &= \alpha v_1+\beta v_2\\ Av_2 &= -\beta v_1+\alpha v_2\end{aligned}\tag{1}$$

4. Let $A$, $\lambda_1=\alpha+i\beta$, $v_1$ and $v_2$ be as in Problem 3.

   (a) Use the result in (1) to deduce that the matrix representation of the linear transformation induced by $A$ relative to the basis $\mathcal{B}=\{v_1,v_2\}$ is
   $$[A]_{\mathcal{B}}^{\mathcal{B}}=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}.$$

   (b) Give an invertible matrix $Q$ such that $Q^{-1}AQ=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$.

5. Construct solutions of the system
   $$\begin{cases}\dot{x} &= \quad ay;\\ \dot{y} &= -bx,\end{cases}$$
   where $a$ and $b$ are positive constants, and sketch the phase portrait.