Solutions to Review Problems for Final Exam

1. The speed, v, of a falling skydiver is modeled by the differential equation

$$m\frac{dv}{dt} = mg - kv^2, \tag{1}$$

where m is the mass of the skydiver, g is the constant acceleration due to gravity near the surface of the earth, and k is the drag coefficient. Note that m, g and k are positive constants.

(a) Give the units of the parameter k.

***Solution**:* Denoting the units of k by $[k]$, we see from the ODE in (1) that

$$\text{mass}\cdot\frac{\text{lnegth}}{\text{time}^2} = [k]\frac{\text{lenth}^2}{\text{time}^2},$$

from which we get that

$$[k] = \frac{\text{mass}}{\text{lenth}};$$

so that, k has units of mass over length. □

(b) Introduce dimensionless variables

$$u = \frac{v}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda} \tag{2}$$

to write the equation in (1) in the dimensionless form

$$\frac{du}{d\tau} = f(u). \tag{3}$$

Express the scaling parameters μ and λ in terms of the original parameters m, g and k.

***Solution**:* Use the definitions of u and τ in (2), and the Chair Rule to compute

$$\begin{aligned}\frac{du}{d\tau} &= \frac{du}{dt}\cdot\frac{d\tau}{dt}\\ &= \frac{\lambda}{\mu}\frac{dv}{dt};\end{aligned}$$

so that, using (1),

$$\frac{du}{d\tau} = \frac{\lambda}{\mu}\left(g - \frac{k}{m}v^2\right)$$

or, using the definition of u (2)

$$\frac{du}{d\tau} = \frac{\lambda g}{\mu} - \frac{\lambda k \mu^2}{m\mu} u^2,$$

or

$$\frac{du}{d\tau} = \frac{\lambda g}{\mu} - \frac{\lambda k \mu}{m} u^2. \tag{4}$$

Observe that, by virtue of the facts that λ has units of time, and μ has units of speed, or length per time, and the answer to part (a) in this problem, the parameter groupings

$$\frac{\lambda g}{\mu} \quad \text{and} \quad \frac{\lambda k \mu}{m}$$

are dimensionless.

Set

$$\frac{\lambda g}{\mu} = 1 \quad \text{and} \quad \frac{\lambda k \mu}{m} = 1, \tag{5}$$

and solve the equations in (5) simultaneously for λ and μ to get

$$\lambda = \sqrt{\frac{m}{gk}} \tag{6}$$

and

$$\mu = \sqrt{\frac{mg}{k}} \tag{7}$$

With the choices of λ and μ in (6) and (7), respectively, the ODE in (4) can be written us

$$\frac{du}{d\tau} = 1 - u^2, \tag{8}$$

which is in the form of equation (3) with

$$f(u) = 1 - u^2, \quad \text{for } u \in \mathbb{R}. \tag{9}$$

□

(c) Sketch the graph of f versus u, find the equilibrium points of the equation in (3), and use Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

***Solution**:* A sketch of the graph of $f(u)$ versus u, where $f(u)$ is as given in (9). is shown in Figure 1.

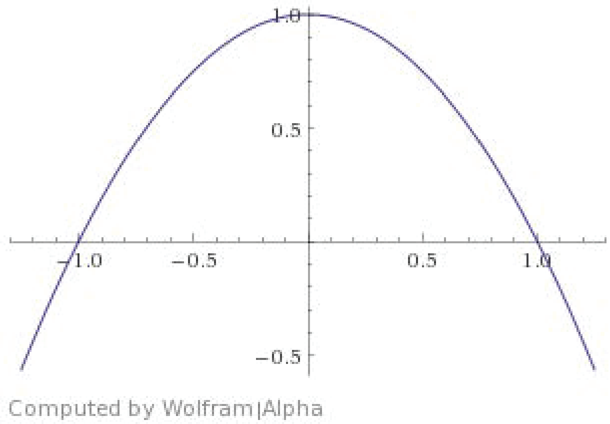

Figure 1: Sketch of $f(u)$ versus u

We see from the sketch in Figure 1 that the ODE in (3), with f as given in (9) has equilibrium points at

$$\overline{u}_1 = -1 \quad \text{and} \quad \overline{u}_2 = 1.$$

We also see from the sketch in Figure 1 that

$$f'(\overline{u}_1) > 0 \quad \text{and} \quad f'(\overline{u}_2) < 0.$$

It then follows from the Principle of Linearized Stability (PLS) that

- $\overline{u}_1 = -1$ is unstable, and
- $\overline{u}_2 = 1$ is asymptotically stable.

□

(d) Sketch the shape of possible solution curves of the equation (3) in the τu–plane for various initial values.

Solution: A sketch of possible solution curves of the ODE in (3), with f as given in (9), was obtained using `pplane` (java version), and is shown in Figure 2. □

(e) Use separation of variables and partial fractions to compute the general solution of the ODE in (3). Use this solution to obtain the general solution of the equation in (1).

Solution: Write the ODE in (8)

$$\frac{du}{d\tau} = -(u^2 - 1),$$

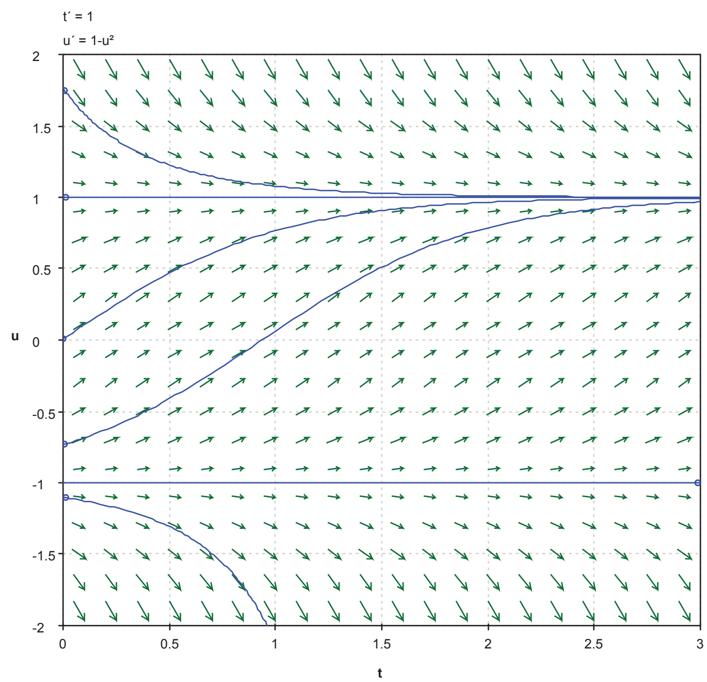

Figure 2: Possible Solutions of (3)

and factor the right–hand side to get

$$\frac{du}{d\tau} = -(u-1)(u+1). \tag{10}$$

Next, use separation of variables and rewrite (10) in differential form

$$\frac{1}{(u-1)(u+1)}\, du = -d\tau,$$

and integrate on both sides to get

$$\int \frac{1}{(u-1)(u+1)}\, du = -\int d\tau,$$

or

$$\int \frac{1}{(u-1)(u+1)}\, du = -\tau + C_1, \tag{11}$$

for some constant of integration C_1.

We can evaluate the integral on the left–hand side of (11) by using partial fractions. Observe that

$$\frac{1}{(u-1)(u+1)} = \frac{1/2}{u-1} + \frac{-1/2}{u+1};$$

so that,

$$\int \frac{1}{(u-1)(u+1)}\,du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C_2,$$

for some constant of integration C_2, or

$$\int \frac{1}{(u-1)(u+1)}\,du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C_2. \tag{12}$$

Combining (11) and (12) then yields

$$\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| = -\tau + C_3,$$

for some constant C_3, or

$$\ln\left|\frac{u-1}{u+1}\right| = -2\tau + C_4, \tag{13}$$

where we have set $c_4 = 2C_3$.

Finally, exponentiating on both sides of (13), and using the continuity of u, we obtain a constant C such that

$$\frac{u-1}{u+1} = Ce^{-2\tau}. \tag{14}$$

Solving the equation in (14) for u the yields

$$u(\tau) = \frac{1+Ce^{-2\tau}}{1-Ce^{-2\tau}}, \tag{15}$$

which is the general solution of (8).

We can obtain the general solution of (1) in view of the change of variables in (2) and the definitions of λ and μ in (6) and (7), respectively. We get

$$v(t) = \sqrt{\frac{mg}{k}} \cdot \frac{1+Ce^{-2t/\sqrt{m/kg}}}{1-Ce^{-2t/\sqrt{m/kg}}}. \tag{16}$$

□

(f) Use the solution of the equation in (1) obtained in the previous part to determine the terminal speed of the skydiver in terms of the original parameters m, g and k.

Solution: With the initial condition $v(0) = 0$, we obtain from (16) that $C = -1$; so that,

$$v(t) = \sqrt{\frac{mg}{k}} \cdot \frac{1 - e^{-2t/\sqrt{m/kg}}}{1 + e^{-2t/\sqrt{m/kg}}}, \quad \text{for all } t \in \mathbb{R}. \tag{17}$$

We obtain from (17) that

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}},$$

which is the terminal speed of the skydiver. □

2. Consider the the nonlinear differential equation

$$\frac{du}{dt} = e^u - 1. \tag{18}$$

Find the equilibrium points of the equations and study their stability.

Solution: Put $f(u) = e^u - 1$, for all $u \in \mathbb{R}$. Then, the ODE in (18) can be written as

$$\frac{du}{dt} = f(u). \tag{19}$$

Note that $f(u) = 0$ if and only if $u = 0$. Thus, $\overline{u} = 0$ is the only equilibrium point of (19).

Next, compute $f'(u) = e^u$; so that, $f'(\overline{u}) = 1 > 0$. Hence, $\overline{u}$ is unstable by the PLS. □

3. In this problem we show how small changes in the coefficients of system of linear equations can affect stability of an equilibrium point that is a center.

   (a) Consider the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Show that $(0, 0)$ a center.

   Solution: Write the system in the form

$$\begin{cases} \dfrac{dx}{dt} = y; \\ \\ \dfrac{dy}{dt} = -x, \end{cases} \tag{20}$$

and observe that the system in (20) has a conserved quantity

$$H(x,y) = x^2 + y^2, \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Note that the level sets,

$$H(x,y) = c,$$

of H are concentric circles about $(0,0)$. Hence, $(0,0)$ is a center. $\square$

(b) Next, consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{21}$$

where $|\varepsilon| \neq 0$ is arbitrarily small. Show that no matter how small $|\varepsilon| \neq 0$ is, the center in part (a) becomes a spiral point. Discuss the stability–type for $\varepsilon > 0$ and for $\varepsilon < 0$.

Solution: Write the system in (21) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{22}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix}. \tag{23}$$

The fundamental matrix of A in (23) is

$$E_A(t) = e^{\varepsilon t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

Thus, any solution, $(x(t), y(t))$, of (22) satisfying the initial condition $(x(0), y(0)) = (x_o, y_o)$ must be of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\varepsilon t} \begin{pmatrix} x_o \cos t + y_o \sin t \\ -x_o \sin t + y_o \cos t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{24}$$

Next, let $r(t)$ denote the distance from $(x(t), y(t)$ to $(0,0)$, for all $t \in \mathbb{R}$; so that,

$$r(t) = \sqrt{(x(t))^2 + (y(t))^2}, \quad \text{for all } t \in \mathbb{R},$$

or

$$(r(t))^2 = (x(t))^2 + (y(t))^2, \quad \text{for all } t \in \mathbb{R}. \tag{25}$$

Next, combine (24) and (25) to compute

$$(r(t))^2 = e^{2\varepsilon t}(x_o \cos t + y_o \sin t)^2 + e^{2\varepsilon t}(-x_o \sin t + y_o \cos t)^2, \quad \text{for } t \in \mathbb{R},$$

from which we get

$$(r(t))^2 = e^{2\varepsilon t}(x_o^2 + y_o^2), \quad \text{for all } t \in \mathbb{R};$$

so that,

$$r(t) = e^{\varepsilon t}\sqrt{x_o^2 + y_o^2}, \quad \text{for all } t \in \mathbb{R}. \tag{26}$$

It follows from (26) that, if $\varepsilon > 0$, then the distance from the trajectory $(x(t), y(t))$ to the origin $(0, 0)$ increases as t increases; thus, if $\varepsilon > 0$, the trajectory $(x(t), y(t))$ spirals away from the origin. In this case, $(0, 0)$ is unstable.

On the other hand, if $\varepsilon < 0$ in (26), we see that $r(t)$ decreases as t increases. Thus, in this case the trajectory will spiral towards the origin. In fact, we compute from (26) that

$$\lim_{t\to\infty} r(t) = 0;$$

so that $(0, 0)$ is asymptotically stable if $\varepsilon < 0$. □

4. Consider the second order, linear, homogeneous differential equation

$$\frac{d^2x}{dt^2} + \mu x = 0, \tag{27}$$

where μ is a real parameter.

   (a) Give the general solution for each of the cases (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

   ***Solution**:* Turn the second order ODE in (27) into a two dimensional system by setting

$$y = \frac{dx}{dt}.$$

   We then have that

$$\frac{dx}{dt} = y$$

and

$$\frac{dy}{dt} = \frac{d^2x}{dt^2},$$

or, in view of (27),

$$\frac{dy}{dt} = -\mu x.$$

We therefore get the system

$$\begin{cases} \dfrac{dx}{dt} = \quad y; \\ \\ \dfrac{dy}{dt} = -\mu x. \end{cases} \tag{28}$$

Write the system in (28) in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{29}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix}. \tag{30}$$

The characteristic polynomial of the matrix A in (30) is

$$p_{_A}(\lambda) = \lambda^2 + \mu. \tag{31}$$

We consider three cases: (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

(i) If $\mu < 0$, the roots of the characteristic polynomial in (31) are the real values $\pm\sqrt{-\mu}$. Thus, in this case, the fundamental matrix of the system in (29) if of the form

$$E_{_A}(t) = Q \begin{pmatrix} e^{-\sqrt{-\mu}\ t} & 0 \\ 0 & e^{\sqrt{-\mu}\ t} \end{pmatrix} Q^{-1}, \qquad \text{for } t \in \mathbb{R},$$

for some change of basis matrix Q. Thus, the first component of the general solution of the system in (29) is a linear combination of the functions

$$x_1(t) = e^{-\sqrt{-\mu}\ t}, \qquad \text{for } t \in \mathbb{R},$$

and

$$x_2(t) = e^{\sqrt{-\mu}\ t}, \qquad \text{for } t \in \mathbb{R},$$

which are linearly independent, since $\mu < 0$. Hence, the general solution of the ODE in (27) is

$$x(t) = c_1 e^{-\sqrt{-\mu}\ t} + c_2 e^{\sqrt{-\mu}\ t}, \quad \text{for } t \in \mathbb{R}, \tag{32}$$

for arbitrary constants c_1 and c_2, for the case $\mu < 0$.

(ii) If $\mu = 0$, then the ODE in (27) becomes

$$\frac{d^2x}{dt^2} = 0,$$

which we can solve by integration. Integrating once, we obtain

$$\frac{dx}{dt} = c_1,$$

where c_1 is a constant of integration. Integrating one more time, we get

$$x(t) = c_1 t + c_2, \quad \text{for } t \in \mathbb{R}, \tag{33}$$

where c_2 is another constant of integration.

(iii) If $\mu > 0$, the roots of the characteristic polynomial in (31) are the complex numbers $\pm i\sqrt{\mu}$. Thus, in this case, the fundamental matrix of the system in (29) if of the form

$$E_{_A}(t) = Q \begin{pmatrix} \cos(\sqrt{\mu}\ t) & -\sin(\sqrt{\mu}\ t) \\ \sin(\sqrt{\mu}\ t) & \cos(\sqrt{\mu}\ t) \end{pmatrix} Q^{-1}, \quad \text{for } t \in \mathbb{R},$$

for some change of basis matrix Q. Thus, the first component of the general solution of the system in (29) is a linear combination of the functions

$$x_1(t) = \cos(\sqrt{\mu}\ t), \quad \text{for } t \in \mathbb{R},$$

and

$$x_2(t) = \sin(\sqrt{\mu}\ t), \quad \text{for } t \in \mathbb{R},$$

which are linearly independent, since $\mu < 0$. Hence, the general solution of the ODE in (27) is

$$x(t) = c_1 \cos(\sqrt{\mu}\ t) + c_2 \sin(\sqrt{\mu}\ t), \quad \text{for } t \in \mathbb{R}, \tag{34}$$

for arbitrary constants c_1 and c_2, for the case $\mu > 0$.

□

(b) For each of the cases (i), (ii) and (iii) in part (a), determine conditions on μ (in any) that will guarantee that the equation in (27) has a nontrivial solution $x\colon \mathbb{R} \to \mathbb{R}$ satisfying $x(0) = 0$ and $x(\pi) = 0$.

Solution*:* We consider three cases: (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

(i) If $\mu < 0$, the general solution of (27) is given by (32); namely,

$$x(t) = c_1 e^{-\sqrt{-\mu}\ t} + c_2 e^{\sqrt{-\mu}\ t}, \quad \text{for } t \in \mathbb{R}, \tag{35}$$

for arbitrary constants c_1 and c_2.
The condition $x(0) = 0$ implies from (35) that

$$c_1 + c_2 = 0, \tag{36}$$

while the condition $x(\pi) = 0$ yields from (35) that

$$c_1 e^{-\pi\sqrt{-\mu}} + c_2 e^{\pi\sqrt{-\mu}} = 0. \tag{37}$$

We can solve the equations in (36) and (37) simultaneously as the system

$$\begin{cases} c_1 + c_2 &= 0; \\ e^{-\pi\sqrt{-\mu}} c_1 + e^{\pi\sqrt{-\mu}} c_2 &= 0, \end{cases}$$

which we can write in matrix form as

$$\begin{pmatrix} 1 & 1 \\ \\ e^{-\pi\sqrt{-\mu}} & e^{\pi\sqrt{-\mu}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{38}$$

The determinant of the 2×2 matrix in (38) is

$$\det \begin{pmatrix} 1 & 1 \\ \\ e^{-\pi\sqrt{-\mu}} & e^{\pi\sqrt{-\mu}} \end{pmatrix} = e^{\pi\sqrt{-\mu}} - e^{-\pi\sqrt{-\mu}}.$$

This determinant is 0 if and only if

$$e^{\pi\sqrt{-\mu}} = e^{-\pi\sqrt{-\mu}}$$

if and only if

$$e^{2\pi\sqrt{-\mu}} = 1$$

if and only if

$$\mu = 0.$$

However, we are assuming that $\mu < 0$. Therefore, we conclude that the determinant of the 2×2 matrix in (38) is nonzero. Consequently, the system in (38) has only the trivial solution

$$c_1 = c_2 = 0.$$

Hence, according to (35), the differential equation in (27), subject to the conditions

$$x(0) = 0 \quad \text{and} \quad x(\pi) = 0$$

has only the trivial solution

$$x(t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

(ii) If $\mu = 0$, the general solution of (27) is given by (33); namely,

$$x(t) = c_1 t + c_2, \quad \text{for } t \in \mathbb{R}, \tag{39}$$

for arbitrary constants c_1 and c_2.
The condition $x(0) = 0$ yields from (39) that $c_2 = 0$; we therefore obtain from (39) that

$$x(t) = c_1 t, \quad \text{for } t \in \mathbb{R}. \tag{40}$$

Next, the condition $x(\pi) = 0$ implies from (40) that

$$c_1 \pi = 0,$$

from which we get that $c_1 = 0$.
Hence, if $\mu = 0$ the differential equation in (27), subject to the conditions

$$x(0) = 0 \quad \text{and} \quad x(\pi) = 0$$

has only the trivial solution

$$x(t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

(iii) If $\mu > 0$, the general solution of (27) is given by (34); namely,

$$x(t) = c_1 \cos(\sqrt{\mu}\, t) + c_2 \sin(\sqrt{\mu}\, t), \quad \text{for } t \in \mathbb{R}, \tag{41}$$

for arbitrary constants c_1 and c_2.
The condition $x(0) = 0$ implies from (41) that

$$c_1 = 0; \tag{42}$$

so that, it follows from (41) that

$$x(t) = c_2 \sin(\sqrt{\mu}\, t), \quad \text{for } t \in \mathbb{R}. \tag{43}$$

The condition $x(\pi) = 0$ yields from (43) that

$$c_2 \sin(\sqrt{\mu}\pi) = 0. \tag{44}$$

Thus, in view (42) and (41), the equation in (27), subject to the conditions

$$x(0) = 0 \quad \text{and} \quad x(\pi) = 0,$$

has a nontrivial solution provided that $c_2 \neq 0$. It then follows from (44) that

$$\sin(\sqrt{\mu}\pi) = 0. \tag{45}$$

It follows from the equation in (45) that

$$\sqrt{\mu}\pi = \pm n\pi, \quad \text{for } n = 1, 2, 3, \ldots \tag{46}$$

Since we are assuming that $\mu > 0$, we obtain from (46) that

$$\mu = n^2, \quad \text{for } n = 1, 2, 3, \ldots \tag{47}$$

Hence, for the values of μ given in (47), the differential equation in (27), subject to the conditions

$$x(0) = 0 \quad \text{and} \quad x(\pi) = 0$$

has nontrivial solutions given by, according to (43),

$$x(t) = C \sin(nt), \quad \text{for all } t, \quad \text{and } n = 1, 2, 3, \ldots,$$

and some constant C.

□

5. Give the general solution of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

   ***Solution**:* Write the system in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{48}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}. \tag{49}$$

The characteristic polynomial of the matrix A in (49) is

$$p_{_A}(\lambda) = \lambda^2 + 6\lambda + 9,$$

which we can factor as

$$p_{_A}(\lambda) = (\lambda + 3)^2.$$

Thus, the matrix A in (49) has only one real eigenvalue

$$\lambda = -3. \tag{50}$$

Next, we compute the eigenspace corresponding to the eigenvalue in (50) and find that the eigenspace is spanned by the single eigenvector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{51}$$

Thus, the matrix A in (49) is not diagonalizable. We can however, turn it into the Jordan canonical form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{52}$$

by a change of basis matrix

$$Q = [\mathrm{v}_1 \quad \mathrm{v}_2], \tag{53}$$

where v_2 is a solution of the system

$$(A - \lambda I)\mathrm{v} = \mathrm{v}_1,$$

where v_1 is given in (50).

We find that we can take

$$\mathrm{v}_2 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}. \tag{54}$$

Thus, using (53) we get that

$$Q = \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix}; \tag{55}$$

so that

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix}. \tag{56}$$

We can then compute the fundamental matrix, $E_A(t)$, of the system in (48),

$$E_A(t) = QE_J(t)Q^{-1}, \quad \text{for } t \in \mathbb{R}, \tag{57}$$

where Q is given in (55), Q^{-1} is given in (56), and $E_J(t)$ is the fundamental matrix corresponding to J in (52),

$$E_J(t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{58}$$

where λ is given in (50).

We get from (57) and (58) that

$$E_A(t) = \begin{pmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{59}$$

The general solution of the system in (48) is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants, and $E_A(t)$ is given in (59). Thus, the general solution of (48) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1(e^{-3t} + 4te^{-3t}) - 4c_2te^{-3t} \\ 4c_1te^{-3t} + c_2(e^{-3t} - 4te^{-3t}) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

□

6. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = x(2 - x - y); \\[2ex] \dfrac{dy}{dt} = y(3 - 2x - y) \end{cases} \tag{60}$$

describes competing species of densities $x \geqslant 0$ and $y \geqslant 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

***Solution**:* The $\dot{x} = 0$–nullclines are the lines

$$x = 0 \quad \text{(the } y\text{–axis)} \quad \text{and} \quad x + y = 2,$$

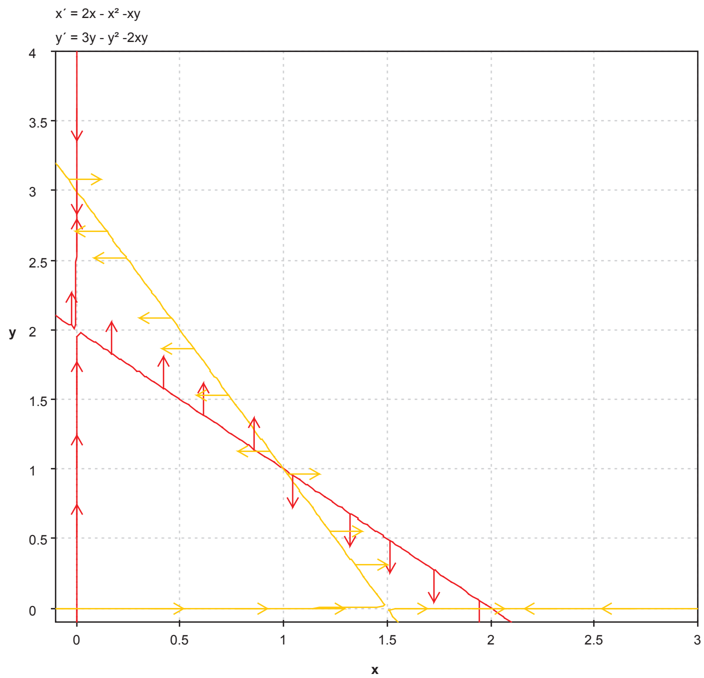

Figure 3: Sketch of Nullclines of the System in (60)

and the $\dot{y} = 0$–nullclines are the lines

$$y = 0 \quad \text{(the } xy\text{–axis)} \qquad \text{and} \qquad 2x + y = 3.$$

These are sketched in Figure 3. We see from the sketch in the figure that there are four equilibrium points:

$$(0, 3), \quad (0, 01), \quad (2, 0) \quad \text{and} \quad (1, 1).$$

To determine the stability properties of the equilibrium points, we look at the derivative of the vector field

$$F(x, y) = \begin{pmatrix} 2x - x^2 - xy \\ 3y - y^2 - 2xy \end{pmatrix}, \qquad \text{for } (x, y) \in \mathbb{R}^2;$$

namely

$$DF(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2y - 2x \end{pmatrix}, \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

We apply the PLS at each of the equilibrium points.

Compute

$$DF(0,3) = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}.$$

Thus, the eigenvalues of the linearization at $(0,3)$ are both negative; hence, $(0,3)$ is a sink, by the PLS.

Next, compute

$$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

which has positive eigenvalues; so that, $(0,0)$ is a source, by the PLS.

$$DF(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix},$$

has negative eigenvalues; thus, $(2,0)$ is a sink, by the PLS.

$$DF(1,1) = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix},$$

which has characteristic polynomial $p(\lambda) = \lambda^2 + 2\lambda - 1$; so that, the eigenvalues eigenvalues of the linearization at $(1,1)$ are $-1 \pm \sqrt{2}$, which are real with opposite signs; thus, $(1,1)$ is a saddle point.

A sketch of possible trajectories obtained using `pplane` is shown in Figure 4.

According to the sketch in Figure 4, the two species will coexist if the initial condition lands at the saddle point $(1,1)$, or any of the two trajectories that converge towards $(1,1)$. The probability of this happening in practice is 0. Hence, it is extremely unlikely that both species will survive. Depending on which region outside those trajectories the initial condition lands, one species will drive the other two extinction; i.e., one of the two sinks; either $(0,3)$ or $(2,0)$. □

7. Consider the two–dimensional, autonomous system

$$\begin{cases} \dfrac{dx}{dt} = (x-y)(1-x^2-y^2); \\[2ex] \dfrac{dy}{dt} = (x+y)(1-x^2-y^2). \end{cases} \tag{61}$$

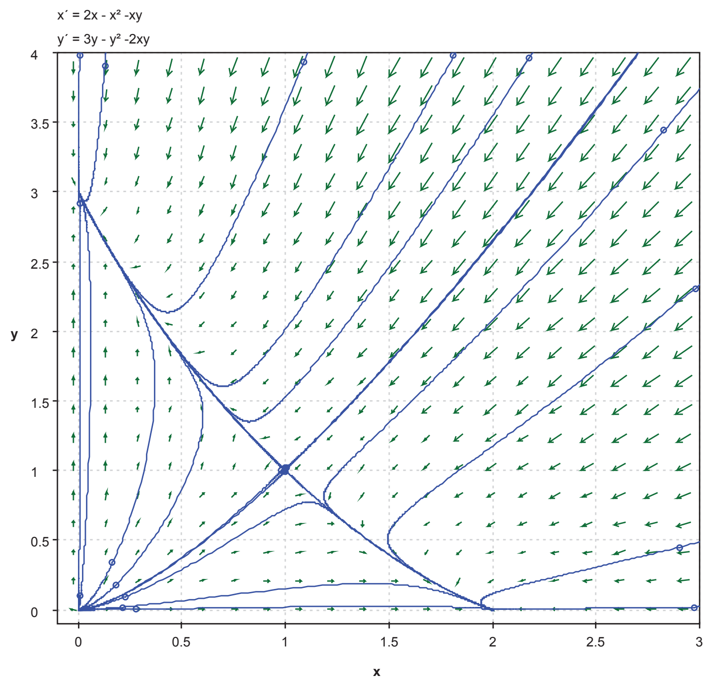

Figure 4: Sketch of Phase Portrait of the System in Problem (60)

(a) Verify that every point in the unit circle,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \tag{62}$$

is an equilibrium point.

Solution: Put

$$F(x, y) = \begin{pmatrix} (x - y)(1 - x^2 - y^2) \\ (x + y)(1 - x^2 - y^2) \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{63}$$

Let $(\overline{x}, \overline{y}) \in C$ then, by the definition of C in (62),

$$\overline{x}^2 + \overline{y}^2 = 1;$$

so that, using the definition of F in (63),

$$F(\overline{x}, \overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows that $(\overline{x}, \overline{y})$ is an equilibrium point of the system (61). $\square$

(b) Show that $(0,0)$ is an isolated equilibrium point of the system.

***Solution**:* Suppose that $(\overline{x},\overline{y})$ is an equilibrium point of the system (61) such that

$$(\overline{x},\overline{y}) \neq (0,0) \tag{64}$$

and

$$\overline{x}^2+\overline{y}^2<1. \tag{65}$$

We then have that

$$F(\overline{x},\overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or, in view of (63),

$$\begin{cases} (\overline{x}-\overline{y})(1-\overline{x}^2-\overline{y}^2) = 0; \\ (\overline{x}+\overline{y})(1-\overline{x}^2-\overline{y}^2) = 0 \end{cases} \tag{66}$$

It follows from (66) and (65) that

$$\begin{cases} \overline{x}-\overline{y} = 0; \\ \overline{x}+\overline{y} = 0. \end{cases} \tag{67}$$

Solving the system in (67) yields that

$$(\overline{x},\overline{y}) = (0,0),$$

which is in direct contradiction with (64). Hence, the only critical point of the system (61) that lies in the disc $x^2+y^2<1$ is the origin $(0,0)$. Consequently, $(0,0)$ is an isolated equilibrium point of the system (61). □

(c) Determine the nature of the stability of $(0,0)$.

***Solution**:* We can apply the Principle of Linearised Stability (PLS). Compute the derivative of the vector field F given in (63) to get

$$DF(x,y) = \begin{pmatrix} (1-x^2-y^2)-2x(x-y) & -(1-x^2-y^2)-2y(x-y) \\ (1-x^2-y^2)-2x(x+y) & (1-x^2-y^2)-2y(x+y) \end{pmatrix},$$

for $(x,y)\in\mathbb{R}^2$. We then have that

$$DF(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{68}$$

The characteristic polynomial of the matrix in (68) is

$$p(\lambda) = \lambda^2-2\lambda+2.$$

The roots of the characteristic polynomial are $1 \pm i$; thus, the eigenvalues of the linearization at $(0,0)$ in (68) are complex with positive real part; hence, by the PLS, $(0,0)$ is a spiral source. □

(d) Let D denote the open unit disc in $\mathbb{R}^2$,

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}. \tag{69}$$

Show that every trajectory that starts at a point $(x_o, y_o) \in D$, such that $(x_o, y_o) \neq (0,0)$, will tend towards C as $t \to \infty$.

***Solution**:* Let $(x(t), y(t))$ denote any trajectory of the system in (61), and let $r(t)$ denote the distance from $(x(t), y(t)$ to $(0,0)$, for all $t \in \mathbb{R}$; so that,

$$r(t) = \sqrt{(x(t))^2 + (y(t))^2}, \quad \text{for all } t \in \mathbb{R},$$

or

$$(r(t))^2 = (x(t))^2 + (y(t))^2, \quad \text{for all } t \in \mathbb{R}. \tag{70}$$

Differentiate with respect to t on both sides of (70) to get

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

where we have used the Chain Rule, or

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}. \tag{71}$$

Combining (71) and (61) we obtain

$$r\frac{dr}{dt} = x(x-y)(1-x^2-y^2) + y(x+y)(1-x^2-y^2),$$

which simplifies to

$$r\frac{dr}{dt} = (x^2+y^2)(1-x^2-y^2).$$

Thus, using (70),

$$r\frac{dr}{dt} = r^2(1-r^2),$$

from which we get that

$$\frac{dr}{dt} = r(1-r^2). \tag{72}$$

Setting

$$f(r) = r(1-r^2), \quad \text{for } r \in \mathbb{R}, \tag{73}$$

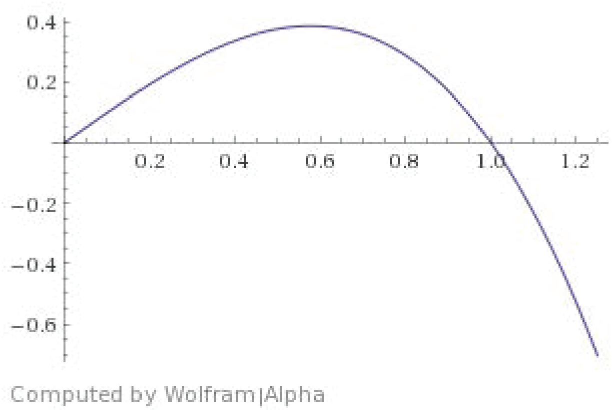

Figure 5: Sketch of $f(r)$ versus r, for $r \geqslant 0$

we can rewrite the equation in (72) as

$$\frac{dr}{dt} = f(r). \tag{74}$$

Figure 5 shows a sketch of the graph of $f(r)$ in (73) versus r, for $r \geqslant 0$. The sketch in the figure shows that the ODE in (74) has equilibrium points at

$$\overline{r}_1 = 0 \quad \text{and} \quad \overline{r}_2 = 1.$$

We also see from the sketch that

$$f'(\overline{r}_1) > 0;$$

so that, $\overline{r}_1 = 0$ is unstable, by the PLS, and

$$f'(\overline{r}_2) < 0,$$

which shows that $\overline{r}_2 = 1$ is asymptotically stable, by the PLS.
The sketch of the graph of $f(r)$ versus r in Figure 5 also shows that $f(r) > 0$ for $0 < r < 1$; so that, if a trajectory, $(x(t), y(t))$, of the system in (61) starts at a point (x_o, y_o) such that

$$0 < x_o^2 + y_o^2 < 1,$$

then, according to (74), $r(t)$ increases towards $\overline{r}_2 = 1$. Furthermore, since $\overline{r}_2 = 1$ us asymptotically stable,

$$\lim_{t\to\infty} r(t) = 1.$$

Hence, the trajectory $(x(t), y(t))$ tends towards C as $t \to \infty$. □

(e) Show that every trajectory that starts at a point $(x_o, y_o) \in \mathbb{R}^2$, such that $x_o^2 + y_o^2 > 1$, will tend towards C as $t \to \infty$.

***Solution**:* The sketch of the graph of $f(r)$ versus r in Figure 5 shows that $f(r) < 0$ for $r > 1$. Thus, if a trajectory, $(x(t), y(t))$, of the system in (61) starts at a point (x_o, y_o) such that

$$x_o^2 + y_o^2 > 1,$$

then, according to (74), $r(t)$ decreases towards $\overline{r}_2 = 1$. Furthermore, since $\overline{r}_2 = 1$ us asymptotically stable,

$$\lim_{t\to\infty} r(t) = 1.$$

Hence, the trajectory $(x(t), y(t))$ tends towards C as $t \to \infty$. □

8. Consider the two–dimensional, autonomous system

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= 4x - x^3. \end{cases} \tag{75}$$

(a) Sketch nullclines, compute equilibrium points, and use the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

***Solution**:* The $\dot{x} = 0$–nullcline is

$$y = 0 \quad \text{(the } x\text{–axis)},$$

and the $\dot{y} = 0$–nullclines are the lines

$$x = -2, \quad x = 0 \quad \text{(the } y\text{–axis)}, \quad \text{and} \quad x = 2.$$

These are sketched in Figure 6. We see from the sketch in Figure 6 the system in (75) has three equilibrium points

$$(-2, 0), \quad (0, 0), \quad \text{and} \quad (2, 0). \tag{76}$$

Next, we looks at the linearization of the system in (75) around each of the equilibrium points in (76).

Put

$$F(x, y) = \begin{pmatrix} y \\ 4x - x^3 \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2,$$

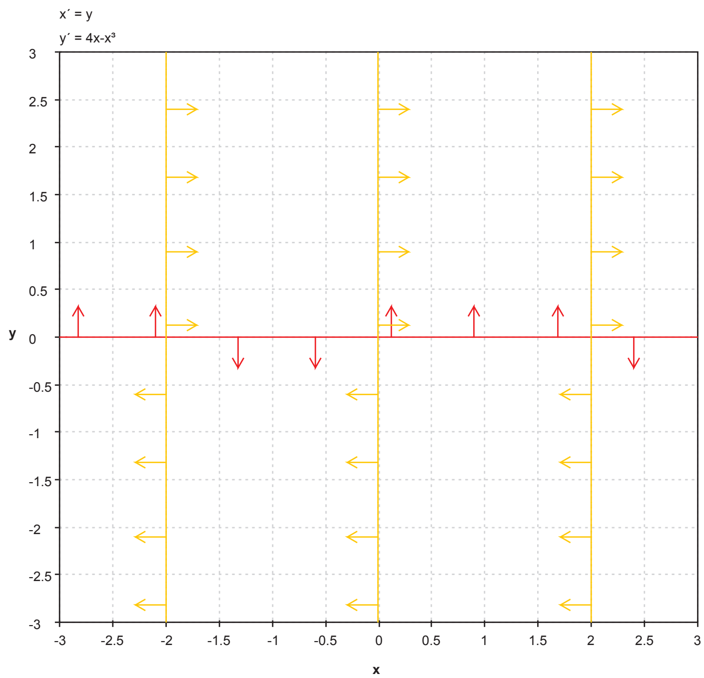

Figure 6: Sketch of Nullclines of the System in (75)

and compute

$$DF(x, y) = \begin{pmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2.$$

The linearization at the equilibrium points $(\pm 2, 0)$ is then,

$$DF(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2,$$

which has characteristic polynomial

$$p(\lambda) = \lambda^2 + 8,$$

which has roots $\pm i2\sqrt{3}$, which are purely imaginary. Hence, the PLS does not apply at the equilibrium points $(\pm 2, 0)$.
The linearization at $(0, 0)$ is

$$DF(0, 0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2,$$

with characteristic polynomial

$$p(\lambda) = \lambda^2 - 4.$$

Consequently, the eigenvalues of $DF(0,0)$ are $\pm\sqrt{2}$. Then, by the PLS, $(0,0)$ is a saddle point of the system in (75). □

(b) Find a conserved quantity for the system.

Solution: Use the Chain Rule and the ODEs in (75) to compute

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4x - x^3}{y}.$$

We can solve the ODE

$$\frac{dy}{dx} = \frac{4x - x^3}{y}$$

by separating variables.

Integrate on both sides of the differential form

$$y\ dy = (4x - x^3)\ dx$$

to get

$$\frac{1}{2}y^2 = 2x^2 - \frac{1}{4}x^4 + C,$$

for some constant of integration C, which can be rewritten as

$$\frac{1}{2}y^2 - 2x^2 + \frac{1}{4}x^4 = C.$$

We therefore get that the system in (75) has the conserved quantity $H(x,y)$ given by

$$H(x,y) = \frac{1}{2}y^2 - 2x^2 + \frac{1}{4}x^4, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{77}$$

□

(c) Discuss the phase–portrait of the system.

Solution: Since the system in (75) has the conserved quantity, $H(x,y)$, given in (77), the trajectories of the system (75) will lie on level curves of H:

$$H(x,y) = C.$$

A few of the level curves of H are sketched in Figure 7. obtained using WolframAlpha®.

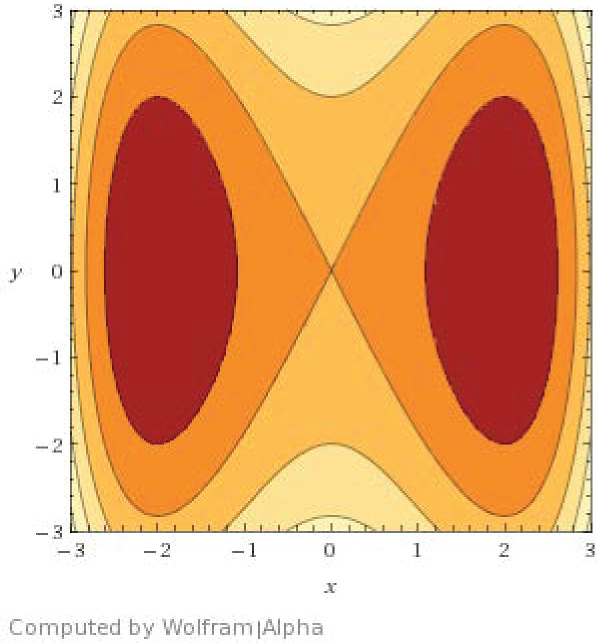

Figure 7: Sketch of level curves of H in (77)

The sketch in Figure 7 shows that there are closed trajectories around the equilibrium points $(\pm 2, 0)$; thus, these points are centers of the system in (75). There sketch also shows that there are orbits emanating from the saddle point at $(0, 0)$ and coming back to it (one of the left and the other on the right); these orbits are called homoclinic orbits.

A sketch of the phase portrait of the system in obtained using the java version of `pplane` is shown in Figure 8.

We see from the sketch in Figure 8 that there are closed orbits surrounding the three equilibrium points. This can also be seen from the contour plot in Figure 7. □

9. Consider the two–dimensional, autonomous system

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2); \\ \dot{y} = x + y - y(x^2 + y^2). \end{cases} \tag{78}$$

(a) Show that $(0, 0)$ is an isolated equilibrium point of the system.

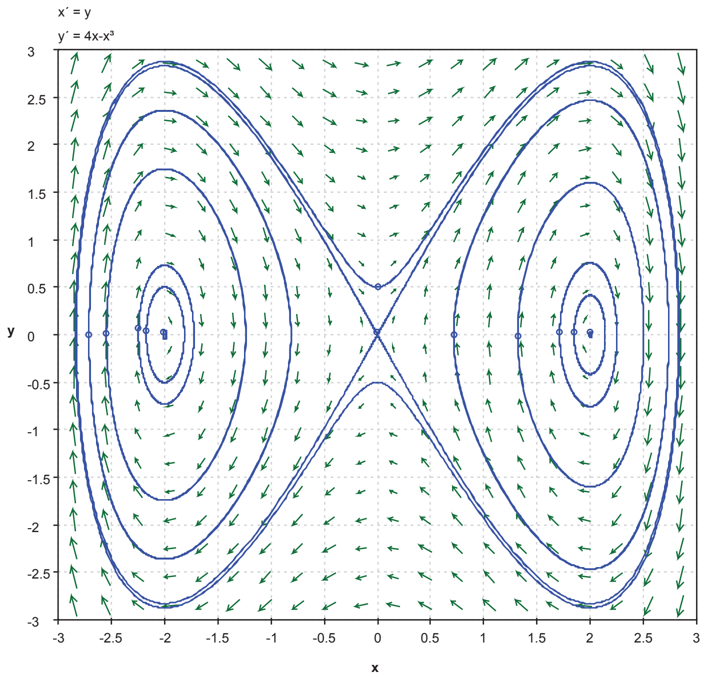

Figure 8: Sketch of phase portrait of the system in (75)

Solution: Put

$$F(x,y) = \begin{pmatrix} x - y - x(x^2+y^2) \\ x + y - y(x^2+y^2) \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{79}$$

and suppose that $(\overline{x},\overline{y})$ is an equilibrium point of the system in (78) such that

$$(\overline{x},\overline{y}) \neq (0,0), \tag{80}$$

and

$$\overline{x}^2 + \overline{y}^2 < 1. \tag{81}$$

We then have that

$$F(\overline{x},\overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or, in view of (79),

$$\begin{cases} \overline{x} - \overline{y} - \overline{x}(\overline{x}^2 + \overline{y}^2) = 0; \\ \overline{x} + \overline{y} - \overline{y}(\overline{x}^2 + \overline{y}^2) = 0. \end{cases} \tag{82}$$

We first argue that

$$\overline{x} \neq 0 \quad \text{and} \quad \overline{y} \neq 0. \tag{83}$$

To see why this is the say, assume, by way of contradiction that $\overline{y} = 0$. it then follows from the second equation in (82) that $\overline{x} = 0$, which contradicts (80). Similarly, if $\overline{x} = 0$, we conclude from the first equation in (82) that $\overline{y} = 0$, which contradicts (80). Hence, the assertion in (83) must be true.

In view of (83) we can multiply the first equation in (82) by $\overline{x}$ and the second by $\overline{y}$ to get

$$\begin{cases} \overline{x}^2 - \overline{x}\,\overline{y} - \overline{x}^2(\overline{x}^2 + \overline{y}^2) = 0; \\ \overline{x}\,\overline{y} + \overline{y}^2 - \overline{y}^2(\overline{x}^2 + \overline{y}^2) = 0. \end{cases} \tag{84}$$

Next, add the two equations in (84) to get

$$\overline{x}^2 + \overline{y}^2 - (\overline{x}^2 + \overline{y}^2)(\overline{x}^2 + \overline{y}^2) = 0,$$

or

$$(\overline{x}^2 + \overline{y}^2)(1 - (\overline{x}^2 + \overline{y}^2)) = 0. \tag{85}$$

In view of (81), we obtain from (85) that

$$\overline{x}^2 + \overline{y}^2 = 0,$$

which is in direct contradiction with (80). We therefore conclude that $(0, 0)$ is the only equilibrium point of the system (78) in the disc $x^2 + y^2 < 1$. Hence, $(0, 0)$ is an isolated equilibrium point of the system in (78). □

(b) Determine the nature of the stability of $(0, 0)$.

***Solution**:* We apply the Principle of Linearized Stability (PLS).

Compute the derivative of the map F given in (79) to obtain

$$DF(x, y) = \begin{pmatrix} 1 - 3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{86}$$

It follows from (86) that

$$DF(0, 0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{87}$$

which has characteristic polynomial

$$p(\lambda) = \lambda^2 - 2\lambda + 2.$$

Thus, the eigenvalues of the matrix $DF(0, 0)$ in (87) are $1 \pm i$. Hence, the eigenvalues of the linearization of the system (78) are complex with positive real part. Consequently, $(0, 0)$ is a spiral source of the system, by the PLS. □

10. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{c}{a + ky} - b; \\ \\ \dfrac{dy}{dt} = \gamma x - \beta, \end{cases} \tag{88}$$

models the time evolution of the interaction of an enzyme of concentration, y, and m–RNA, of concentration x, in a process of protein synthesis. The parameters a, b, c, k, α and β are assumed to be positive. This model was proposed by Brian C. Goodwin in 1965 (*Oscillatory behavior in enzymatic control processes,* in Advances in Enzyme Regulation, Volume 3, 1965, Pages 425–428, IN1–IN2, 429430, IN3–IN6, 431–437).

   (a) Sketch the nullclines, find all equilibrium points, and apply the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

   (b) Find a conserved quantity for the the system.

   (c) Discuss the phase–portrait of the system.