Assignment #4

Due on Friday, February 16, 2018

Read Section 5.1.2 on *Fourier Series Expansions* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions
In problems 1 and 2, we use the Dirichlet kernel,

$$D_N(z) = \frac{\sin\left[\left(N+\frac{1}{2}\right)z\right]}{2\sin(z/2)}, \quad \text{for } z \neq 0z \neq 0 \text{ and } z \in [-\pi,\pi], \tag{1}$$

to evaluate the improper integral $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt$.
Note that the definition of D_N given in (1) is the same as the one given in the lecture notes with $L=\pi$. Recall that this is the same as

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nz), \quad \text{for } z \in [-\pi,\pi]. \tag{2}$$

Do the following problems

1. **The Dirichlet Integral**. Define $g(x) = \dfrac{1}{x} - \dfrac{1}{2\sin(x/2)}$, for $x \neq 0$.

   (a) Use L'Hospital's Rule to compute $\lim_{x\to 0} g(x)$. Use this result to define $g(x)$ at $x=0$ so that g is continuous on $[-\pi,\pi]$, and hence absolutely integrable on $[-\pi,\pi]$.

   (b) Show how to define $g'(0)$ so that the function g defined in part (a) above becomes a C^1 function defined on $[-\pi,\pi]$.

   (c) With the definition of g given in part (a) above, explain why

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} g(x) \sin\left[\left(N+\frac{1}{2}\right)x\right] dx = 0. \tag{3}$$

2. **The Dirichlet Integral (Continued)**. Note that the improper integral

$$\int_0^\infty \frac{\sin t}{t}\, dt$$

can be evaluated as

$$\int_0^\infty \frac{\sin t}{t}\, dt = \lim_{N\to\infty} \int_0^{(N+1/2)\pi} \frac{\sin t}{t}\, dt. \tag{4}$$

   (a) Make the change of variables $t = \left(N + \dfrac{1}{2}\right) z$ in the definite integral on the right–hand side of (4) to rewrite the integral as

$$\int_0^{(N+1/2)\pi} \frac{\sin t}{t}\, dt = \frac{1}{2}\int_{-\pi}^{\pi} g(z) \sin\left[\left(N+\frac{1}{2}\right) z\right] dx + \frac{1}{2}\int_{-\pi}^{\pi} D_{_N}(z)\, dz, \tag{5}$$

   where g is the function defined in part (a) of Problem 1.

   (b) Use (4), (5) and (3) to evaluate the integral $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt$.

3. **The Sine Integral Function**. The sine integral function, $\mathrm{Si}\colon \mathbb{R} \to \mathbb{R}$, is defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt, \qquad \text{for } x \in \mathbb{R}.$$

   (a) Use graphing software to sketch graph of $y = \mathrm{Si}(x)$.

   (b) Use the fact that $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ to prove that the sine integral function is bounded. That is, there exists $M > 0$ such that

$$|\mathrm{Si}(x)| \leqslant M, \qquad \text{for all } x \in \mathbb{R}.$$

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a functions that is integrable on bounded, closed intervals of $\mathbb{R}$, and define

$$F(x) = \int_0^x f(t)\, dt, \qquad \text{for all } x \in \mathbb{R}.$$

   (a) Show that, if f is even, then F is odd.

   (b) Show that, if f is odd, then f is even.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable, $2L$–periodic function whose derivative, f', is absolutely integrable on $[-L, L]$; that is,

$$\int_{-L}^{L} |f'(x)|\; dx < \infty.$$

Let a_o, a_n, b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f, and a_o', a_n', b_n', for $n \in \mathbb{N}$, denote the Fourier coefficients of f'.

(a) Show that $a_o' = 0$.

(b) Derive the identities

$$a_n' = \frac{n\pi}{L} b_n, \qquad \text{for } n = 1, 2, 3, \ldots$$

and

$$b_n' = -\frac{n\pi}{L} a_n, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Deduce that the Fourier coefficients of f' can be obtained by term–by–term differentiation of the Fourier series expansion for f.

(c) Show that

$$\lim_{n\to\infty} (na_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} (nb_n) = 0.$$

Give an interpretation for this result.