Assignment #6

Due on Monday, April 2, 2018

Read Section 5.3 on *The Dirichlet Problem for the Unit Disc* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions

In Section 5.3.6 of the class lecture notes at `http://pages.pomona.edu/~ajr04747/`, we showed that, for any given $g \in C(\partial D_1, \mathbb{R})$, where $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is the open, unit disc in $\mathbb{R}^2$, there exists a function $u \in C^2(D_1, \mathbb{R}) \cap C(\overline{D}_1, \mathbb{R})$ that solves the BVP

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_1; \\ u(x, y) = g(x, y), & \text{for } (x, y) \in \partial D_1, \end{cases}$$

Indeed, u is given by the Poisson integral representation

$$u(x, y) = \begin{cases} \displaystyle\oint_{\partial D_1} P((x, y), (\xi, \eta)) g((\xi, \eta))\, ds_1, & \text{for } (x, y) \in D_1; \\ \\ g(x, y), & \text{for } (x, y) \in \partial D_1, \end{cases} \tag{1}$$

where $P((x, y), (\xi, \eta))$, for $(x, y) \in D_1$ and $(\xi, \eta) \in \partial D_1$, is the Poisson kernel for the unit disc, D_1, given by

$$P((x, y), (\xi, \eta)) = \frac{1}{2\pi} \, \frac{1 - |(x, y)|^2}{|(x, y) - (\xi, \eta)|^2}, \tag{2}$$

for $(x, y) \in D_1$ and $(\xi, \eta) \in \partial D_1$, where $|\cdot|$ denotes the Euclidean norm of vectors in $\mathbb{R}^2$.

In this problem set we will see how to construct a solution of the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega; \\ u(x, y) = g(x, y), & \text{for } (x, y) \in \partial\Omega, \end{cases}$$

where $g \in C(\partial\Omega, \mathbb{R})$ is given and Ω is either

$$D_R(0, 0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\},$$

for $R > 0$, or

$$D_R(x_o, y_o) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_o)^2 + (y - y_o)^2 < R^2\},$$

for given $(x_o, y_o) \in \mathbb{R}^2$.

Do the following problems.

1. Let Ω denote an open, bounded subset of $\mathbb{R}^2$ that contains the origin $(0,0)$. For $\lambda > 0$, define
$$\Omega_\lambda = \{(x,y) \in \mathbb{R}^2 \mid (\lambda x, \lambda y) \in \Omega\}.$$
Given $v \in C^2(\Omega, \mathbb{R})$, define $u\colon \Omega_\lambda \to \mathbb{R}$ by
$$u(x,y) = v(\lambda x, \lambda y), \qquad \text{for } (x,y) \in \Omega_\lambda. \tag{3}$$
   (a) Show that Ω_λ is an open and bounded subset of $\mathbb{R}^2$.
   (b) Let u be as defined in (3). Verify that
   $$u_{xx} + u_{yy} = \lambda^2 (v_{xx} + v_{yy}).$$
   Deduce therefore that, if v is harmonic in Ω, then u is harmonic in Ω_λ.

2. For $R > 0$, define
$$D_R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}$$
and
$$\overline{D}_R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant R^2\},$$
the closure of D_R. Suppose that $u \in C^2(D_R, \mathbb{R}) \cap C(\overline{D}_R, \mathbb{R})$ solves the Dirichlet problem
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_R; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial D_R, \end{cases} \tag{4}$$
where $g \in C(\partial D_R, \mathbb{R})$ is given.
Define $v\colon \overline{D}_1 \to \mathbb{R}$ by
$$v(x,y) = u(Rx, Ry), \qquad \text{for all } (x,y) \in \overline{D}_1. \tag{5}$$
   (a) Verify that the function v defined in (5) solves the Dirichelet problem
   $$\begin{cases} v_{xx} + v_{yy} = 0 & \text{in } D_1; \\ v(x,y) = g(Rx, Ry), & \text{for } (x,y) \in \partial D_1. \end{cases} \tag{6}$$
   (b) Use the Poisson Integral representation for the solution of (6) given in (1) and (2) to obtain an integral representation for a solution, u, of the Dirichlet problem (4).
   Suggestion: Let u be a solution of (4); then, for any $(x,y) \in D_R$, write
   $$u(x,y) = u\left(R\frac{x}{R}, R\frac{y}{R}\right) = v\left(\frac{x}{R}, \frac{y}{R}\right).$$

(c) Give a formula for the Poisson kernel for D_R.

3. Let Ω denote an open, bounded subset of $\mathbb{R}^2$ that contains the origin $(0,0)$. For $(x_o, y_o) \in \mathbb{R}^2$, define

$$\Omega_{(x_o,y_o)} = \{(x,y) + (x_o,y_o) \in \mathbb{R}^2 \mid (x,y) \in \Omega\}.$$

   (a) Show that $\Omega_{(x_o,y_o)}$ is an open and bounded subset of $\mathbb{R}^2$.

   (b) Let $v \in C^2(\Omega, \mathbb{R})$ and define

$$u(x,y) = v(x - x_o, y - y_o), \quad \text{for } (x,y) \in \Omega_{(x_o,y_o)}.$$

   Show that, if v is harmonic in Ω, then u is harmonic in $\Omega_{(x_o,y_o)}$

4. For $R > 0$ and $(x_o, y_o) \in \mathbb{R}^2$, define

$$\Omega = D_R(x_o,y_o) = \{(x,y) \in \mathbb{R}^2 \mid (x-x_o)^2 + (y-y_o)^2 < R^2\},$$

the disc of radius R centered at (x_o, y_o).

   (a) Use the result of part (b) in Problem 2 and the result of part (b) in Problem 3 to construct a solution of the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial\Omega, \end{cases}$$

   for a given $g \in C(\partial\Omega, \mathbb{R})$.

   (b) Give a formula for the Poisson kernel of $D_R(x_o, y_o)$.

5. **The Mean–Value Property for Harmonic Functions**. Let Ω denote an open subset of $\mathbb{R}^2$ and $(x_o, y_o) \in \Omega$. Let $r > 0$ be such that

$$\overline{D_r(x_o,y_o)} = \{(x,y) \in \mathbb{R}^2 \mid (x-x_o)^2 + (y-y_o)^2 \leqslant r^2\} \subset \Omega.$$

Assume that $u \in C^2(\Omega, \mathbb{R})$ is harmonic in Ω.

(a) Use the result Problem 4 to show that

$$u(x_o.y_o) = \frac{1}{2\pi r}\oint_{\partial D_r(x_o,y_o)} u(\xi,\eta)\ ds_r;$$

that is, if $u \in C^2(\Omega, \mathbb{R})$ is harmonic in Ω, then $u(x_o, y_o)$ is the average value of u over any circle centered at (x_o, y_o) and contained in Ω.

(b) Use the result from part (a) above to show that

$$u(x_o.y_o) = \frac{1}{\pi R^2}\iint_{D_R(x_o,y_o)} u(x,y)\ dxdy,$$

for any $R > 0$ such that $\overline{D_R(x_o, y_o)} \subset \Omega$.