Assignment #7

Due on Friday, April 13, 2018

Read Section 5.4 on *Green's Function* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 1.5.2 on *Green's Function Method*, pp. 28–33, in *Introduction to Partial Differential Equations and Hilbert Space Methods* by Karl E. Gustafson.

Background and Definitions
The Support of a Function. Given a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, the support of φ, denoted $\text{Supp}(\varphi)$, is the closure of the set where φ is nonzero; that is,

$$\text{Supp}(\varphi) = \overline{\{(x,y) \in \mathbb{R}^2 \mid \varphi(x,y) \neq 0\}}.$$

If $\text{Supp}(\varphi)$ is also bounded, then it is compact, and we say that φ has **compact support**. Let Ω denote an open subset of $\mathbb{R}^2$. We denote by $C_c^\infty(\Omega)$ the space of real–valued, C^∞ functions, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, that have compact support contained in Ω.

Do the following problems.

1. Let Ω be an open subset of $\mathbb{R}^2$ and $u \in C(\Omega, \mathbb{R})$. Let $(x_o, y_o) \in \Omega$ and $r > 0$ be such that $\overline{D}_r(x_o, y_o) \subset \Omega$. Show that there exists $\omega_r \in [-\pi, \pi]$ such that

$$\oint_{\partial D_r(x_o,y_o)} u(x,y)\ ds = 2\pi r u(x_o + r\cos(\omega_r), y_o + r\sin(\omega_r)).$$

   Deduce that $\displaystyle\lim_{r\to 0^+} \frac{1}{2\pi r} \oint_{\partial D_r(x_o,y_o)} u(x,y)\ ds = u(x_o, y_o)$.

2. **Locally Integrable Functions**. Let $\mathcal{U}$ denote an open subset of $\mathbb{R}^2$. A function $w\colon \mathcal{U} \to \mathbb{R} \cup \{-\infty, +\infty\}$ (that is, $|w|$ could be infinite at a point, or points, in $\mathcal{U}$) is said to be locally integrable in $\mathcal{U}$ if and only if, for every disc, D, such that $\overline{D} \subset \mathcal{U}$,

$$\iint_{\overline{D}} |w|\ dxdy < \infty.$$

   Define $W\colon \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$W(x,y) = \begin{cases} -\dfrac{1}{2\pi} \ln |(x,y)|, & \text{if } (x,y) \neq (0,0); \\ \\ +\infty, & \text{if } (x,y) = (0,0). \end{cases} \tag{1}$$

   Verify that the function W defined in (1) is locally integrable in $\mathbb{R}^2$.

3. **Integration by Parts in Two Dimensions**. Let $\mathcal{U}$ denote an open subset of $\mathbb{R}^2$ and Ω a bounded subset of $\mathcal{U}$ with piecewise C^1 boundary, $\partial\Omega$, and such that $\overline{\Omega} \subset \mathcal{U}$.

   (a) Let $u, v \in C^1(\mathcal{U}, \mathbb{R})$. Use the divergence theorem to derive the following integration by parts formulas in $\mathbb{R}^2$.

   $$\iint_\Omega u\frac{\partial v}{\partial x}\ dxdy = \oint_{\partial\Omega} uvn_1\ ds - \iint_\Omega \frac{\partial u}{\partial x}v\ dxdy,$$

   and

   $$\iint_\Omega u\frac{\partial v}{\partial y}\ dxdy = \oint_{\partial\Omega} uvn_2\ ds - \iint_\Omega \frac{\partial u}{\partial y}v\ dxdy,$$

   where n_1 and n_2 are the components of the outward, unit normal vector $\widehat{n} = (n_1, n_2)$ on the boundary, $\partial\Omega$, of Ω.

   (b) Show that

   $$\iint_\Omega u\frac{\partial \varphi}{\partial x}\ dxdy = -\iint_\Omega \frac{\partial u}{\partial x}\varphi\ dxdy, \qquad \text{for every } \varphi \in C_c^\infty(\Omega),$$

   and

   $$\iint_\Omega u\frac{\partial \varphi}{\partial \varphi}\ dxdy = -\iint_\Omega \frac{\partial u}{\partial y}\varphi\ dxdy, \qquad \text{for every } \varphi \in C_c^\infty(\Omega).$$

4. **Weak Derivatives**. Let Ω denote an open subset of $\mathbb{R}^2$, and let $w\colon \Omega \to \mathbb{R}\cup\{-\infty, +\infty\}$ be a locally integrable function. Suppose that there exist locally integrable functions v_1 and v_2 such that

   $$\iint_\Omega w\frac{\partial \varphi}{\partial x}\ dxdy = -\iint_\Omega v_1\varphi\ dxdy, \qquad \text{for all } \varphi \in C_c^\infty(\Omega),$$

   and

   $$\iint_\Omega w\frac{\partial \varphi}{\partial y}\ dxdy = -\iint_\Omega v_2\varphi\ dxdy, \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

   We then say that v_1 and v_2 are **weak partial derivatives** of w. We denote them by $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$, respectively, even though the functions w might not have partial derivatives in the usual sense of Multivariable Calculus.

   (a) Let $u \in C^1(\Omega, \mathbb{R})$. Verify that u has weak partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

(b) Suppose that a locally integrable function $w\colon \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ has second order, weak partial derivatives. Verify that

$$\iint_\Omega w(\Delta\varphi)\ dxdy = \iint_\Omega (\Delta w)\varphi\ dxdy, \qquad \text{for all } \varphi \in C_c^\infty(\Omega),$$

where Δw denotes the weak Laplacian of w.

5. Let $\mathcal{U}$ denote an open subset of $\mathbb{R}^2$ and Ω a bounded, connected, open subset of $\mathcal{U}$ satisfying $\overline{\Omega} \subset \mathcal{U}$, and having a piecewise C^1 boundary, $\partial\Omega$.

For $(x, y), (\xi, \eta) \in \mathbb{R}^2$ define

$$W((x, y), (\xi, \eta)) = -\frac{1}{2\pi} \ln |(x, y) - (\xi, \eta)|, \quad \text{provided that } (x, y) \neq (\xi, \eta). \tag{2}$$

Let $u \in C^2(\mathcal{U}, \mathbb{R})$. In the class lecture notes we derived the following representation formula

$$\begin{aligned} u(x, y) \ &= \ -\iint_\Omega W((x, y), (\xi, \eta))\Delta u(\xi, \eta)\ d\xi d\eta \\ &\quad + \oint_{\partial\Omega} \left(W((x, y), (\xi, \eta)\frac{\partial u}{\partial n} - u\frac{\partial W((x, y), (\xi, \eta))}{\partial n} \right)\ ds, \end{aligned} \tag{3}$$

where W is defined in (2) and we have written u for $u(\xi, \eta)$ and $\dfrac{\partial u}{\partial n}$ for $\dfrac{\partial}{\partial n}[u(\xi, \eta)]$ in the line integral in (3).

(a) Use the representation formula in (3) to show that

$$\iint_\Omega W((x, y), (\xi, \eta))(-\Delta\varphi(\xi, \eta))\ d\xi d\eta = \varphi(x, y), \tag{4}$$

for all $\varphi \in C_c^\infty(\Omega)$, where W is as defined in (2).

(b) Use the result of part (b) in Problem 4 to deduce from (4) that

$$\iint_\Omega (-\Delta W)\varphi\ d\xi d\eta = \varphi(x, y), \qquad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{5}$$

where ΔW denotes the weak Laplacian of the function W defined in (2) with respect to the variables ξ and η.

The right–hand side of (5) is the definition of the Dirac distribution, $\delta_{(x,y)}$, in the sense that

$$\iint_\Omega \delta_{(x,y)}(\xi,\eta)\varphi(\xi,\eta)\; d\xi d\eta = \varphi(x,y), \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In this sense, the equation in (5) can be written as

$$\iint_\Omega (-\Delta W)\varphi\; d\xi d\eta = \iint_\Omega \delta_{(x,y)}\varphi(\xi,\eta)\; d\xi d\eta, \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \quad (6)$$

The equation in (6) gives meaning to the statement that W is the weak solution of the equation

$$-\Delta W = \delta_{(x,y)}.$$

This is what it means for the function W to be the fundamental solution of Poisson's equation.