Assignment #1

Due on Friday, February 1, 2019

Read Chapter 1, *Introduction,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Chapter 2, *Fundamental Existence Theory,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^N$, and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. The system
$$\frac{dx}{dt} = F(x) \tag{1}$$
is said to be autonomous because the vector field, F, does not depend explicitly on the "time" variable, t.

   Suppose that $u\colon J \to U$ is a C^1 curve defined on an open interval, J, which solves the differential equation in (1); that is,
$$u'(t) = F(u(t)), \quad \text{for all } t \in J.$$

   For a given real constant, c, define the interval J_c to be
$$J_c = \{t \in \mathbb{R} \mid t + c \in J\}.$$

   Define a curve $v\colon J_c \to U$ by $v(t) = u(t+c)$ for all $t \in J_c$.

   Verify that v is also a solution of (1); that is, show that v satisfies
$$v'(t) = F(v(t)), \quad \text{for all } t \in J_c.$$

   Suggestion: Apply the Chain Rule.

2. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by
$$F(x) = \begin{cases} 0 & \text{if } x \leqslant 0; \\ \sqrt{x} & \text{if } x > 0. \end{cases}$$

   (a) Verify that the function $u\colon \mathbb{R} \to \mathbb{R}$ given by
$$u(t) = \begin{cases} 0 & \text{if } t \leqslant 0; \\ \dfrac{t^2}{4} & \text{if } t > 0, \end{cases}$$

solves the initial value problem (IVP)

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = 0. \end{cases} \tag{2}$$

(b) Give another solution to the IVP (2).

(c) Use the result of Problem 1 to come up with infinitely many solutions of the IVP (2).

3. Let U denote an open subset of $\mathbb{R}^N$ which contains the zero vector, 0, and let J denote an open interval of real numbers containing 0. Assume that $F\colon U \to \mathbb{R}^N$ is a C^1 vector field satisfying $F(0) = 0$. Show that if $u\colon J \to U$ is a solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = 0, \end{cases}$$

then u must be identically 0 on J.

Suggestion: Apply the local existence and uniqueness theorem for ordinary differential equations.

4. Let U denote an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. Let $p_o \in U$ and assume that $u\colon J \to U$ solves the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(t_o) = p_o, \end{cases}$$

where J is an open interval containing t_o. Show that u is a C^2 function; that is, u has a a continuous second derivative, u'', defined on J.

Write down the second order differential equation that u satisfies and the corresponding initial value problem.

Suggestion: Apply the Chain Rule.

5. *(Gromwall's Lemma)* Let u and v denote continuous, real valued functions defined in the closed interval $[a, b]$. Assume that

$$|u(t)| \leqslant C + \int_a^t |u(\tau)|\, |v(\tau)|\, \mathrm{d}\tau, \qquad \text{for all } t \in [a, b].$$

(a) Prove that

$$|u(t)| \leqslant Ce^{V(t)}, \quad \text{for all } t \in [a,b], \tag{3}$$

where

$$V(t) = \int_a^t |v(\tau)|\, \mathrm{d}\tau, \quad \text{for all } t \in [a,b].$$

The inequality in (3) is usually referred to as Gronwall's inequality.

(b) Apply the result in (3) of the previous part to the situation in which $v(t) = K$, for all $t \in [a,b]$, where K is a positive constant.

Suggestion: Define the real value function, $g\colon [a,b] \to \mathbb{R}$,

$$g(t) = C + \int_a^t |u(\tau)|\, |v(\tau)|\, \mathrm{d}\tau, \quad \text{for all } t \in [a,b].$$

Then, use the Fundamental Theorem of Calculus to show that g is differentiable on (a,b), and to derive a differential inequality satisfied by g.