Assignment #4

Due on Monday, March 4, 2019

Read Chapter 3, on *Flows,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

Let U denote an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector filed. Define $\mathcal{D} \subset \mathbb{R}\times U$ by

$$\mathcal{D} = \{(t,p)\in\mathbb{R}\times U \mid t\in J_p\},$$

where J_p denotes the maximal interval of existence for the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p. \end{cases} \tag{1}$$

- *(Flow Maps)* The flow map $\theta\colon \mathcal{D}\to U$ of the vector field F is defined to be

  $$\theta(t,p) = u_p(t), \quad \text{for all } (t,p)\in\mathcal{D},$$

  where $u_p\colon J_p \to U$ is the solution to IVP (1)

- *(Orbits)* Given $p\in U$, the orbit, or trajectory, of the flow, θ, through p is the set, γ_p, defined by

  $$\gamma_p = \{x\in U \mid x = \theta(t,p) \text{ for some } t\in J_p\};$$

  in other words, γ_p is the image of the solution, $u_p\colon J_p\to U$, of the IVP (1). Thus, γ_p is a C^1 curve in U parametrized by $u_p\colon J_p \to U$. This is what we have been calling the integral curve of F through p.

- Assume that $\theta(t,p)$ is defined for all $t\in\mathbb{R}$ and all $p\in U$. For each $t\in\mathbb{R}$ define $\theta_t\colon U\to U$ by $\theta_t(p) = \theta(t,p)$, for all $p\in U$.

- *(Invariant Sets)* A subset A of U is said to be invariant under the flow θ if for every $p\in A$, $\theta(t,p)\in A$ for all $t\in J_p$.

- *(Fixed Points or Singular Points)* A point p in U is said to be a fixed point of the flow θ if $\theta(t,p) = p$ for all $t\in J_p$. Fixed points are also referred to as singular points, or equilibrium points.

Do the following problems

1. Let $\theta(t,p)$ denote the flow map of the C^1 vector field $F\colon U \to \mathbb{R}$. Assume that $\theta(t,p)$ is defined for all $t \in \mathbb{R}$ and all $p \in U$. Prove that:

   (a) θ_0 is the identity map on U; that is, $\theta(0,p) = p$ for all $p \in U$.

   (b) For any t and s in $\mathbb{R}$, $\theta_{t+s} = \theta_t \circ \theta_s$.

   (c) Deduce from (a) and (b) above that θ_t is invertible for any $t \in \mathbb{R}$.

2. Let γ_p denote the orbit through p in U of the flow θ. Prove that

$$q \in \gamma_p \Rightarrow \gamma_p = \gamma_q.$$

   Give an interpretation of this result.

3. Let γ_p and γ_q denote the orbits through p and q in U, respectively, of the flow θ. Prove that

$$\gamma_p \cap \gamma_q \neq \emptyset \Rightarrow \gamma_p = \gamma_q.$$

   Deduce therefore that distinct orbits do not intersect.

   Suggestion: Let $p_o \in \gamma_p \cap \gamma_q$; then, there exist $t_1 \in J_p$ and $t_2 \in J_q$ such that

$$u_p(t_1) = u_q(t_2) = p_o.$$

4. Let γ_p denote the orbit through p in U of the flow θ. Prove that γ_p is invariant under the flow.

5. Let $p^* \in U$ denote a fixed point of the flow θ. Prove that

   (a) $F(p^*) = 0$, the zero–vector in $\mathbb{R}^N$. The point p^* is also referred to as an equilibrium point of the differential equation

$$\frac{dx}{dt} = F(x). \tag{2}$$

   (b) Let $p^* \in U$ and suppose that $F(p^*) = 0$. Compute γ_{p^*}.

   (c) Let A denote the set of equilibrium points of the equation (2). Prove that A is invariant under the flow θ.