## Assignment #6

**Due on Monday, April 8, 2019**

**Read** Chapter 4, on *Continuous Dynamical Systems* , in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Do** the following problems

1. For real numbers $a$ and $b$ with $a^2 + b^2 \neq 0$, let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

   (a) Explain why the dynamical system, $\theta(t, p, q)$, for $(t, p, q) \in \mathbb{R}^3$ corresponding to the field $F$ exists.

   (b) Prove that $(0, 0)$ is the only equilibrium point of the field $F$.

   (c) Define $V(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$. Given $(p, q) \in \mathbb{R}^2$ with $(p, q) \neq (0, 0)$, define

$$v(t) = V(\theta(t, p, q)), \quad \text{for all } t \in \mathbb{R};$$

   that is, the function $v$ gives the values of $V$ on the orbit $\gamma_{(p,q)}$.
   Compute $v'(t)$ and deduce from your result that if $a < 0$, then $V$ decreases on $\gamma_{(p,q)}$ as $t$ increases. What happens when $a > 0$.

   (d) Compute the $\omega$–limit sets of $\gamma_{(p,q)}$, for $(p, q) \neq (0, 0)$, in the cases $a < 0$ and $a > 0$.

   (e) Compute the $\alpha$–limit sets of $\gamma_{(p,q)}$, for $(p, q) \neq (0, 0)$, in the cases $a < 0$ and $a > 0$.

2. Assume that $r = r(t)$ and $\theta = \theta(t)$ are differentiable functions of $t \in \mathbb{R}$, and define $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$ for all $t \in \mathbb{R}$. Verify that

$$\begin{aligned} \frac{dr}{dt} &= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta \\ \frac{d\theta}{dt} &= \frac{1}{r}\frac{dy}{dt}\cos\theta - \frac{1}{r}\frac{dx}{dt}\sin\theta. \end{aligned} \tag{1}$$

3. Use the transformation equations (1) derived in the previous problem to transform the system

$$\begin{cases} \dfrac{dx}{dt} &= ax - by; \\ \\ \dfrac{dy}{dt} &= bx + ay. \end{cases} \tag{2}$$

into a system involving $r$ and $\theta$.

   (a) Solve the system for $r$ and $\theta$.

   (b) Based on your formulas for $r$ and $\theta$, write down the general solution to the system (2)

   (c) Use your result in the previous part to obtain the dynamical system, $\theta(t, p, q)$, for $(t, p, q) \in \mathbb{R}^3$, for the system in (2). Explain why this is the same system as the one mentioned in Part (a) of Problem 1.

4. Assume that $b > 0$ and $a \neq 0$ in the two–dimensional system (2).

   (a) Based on your solution to the previous problem in terms of $r$ and $\theta$, sketch a possible non–trivial orbit of the system. Compute the $\alpha$–limit set of the orbit. What is the $\omega$–limit set of the orbit?

   (b) Assume that $a < 0$ in the two–dimensional system (2). Based on your solution in terms of $r$ and $\theta$ resulting from the transformation equations (1), sketch a possible non–trivial orbit of the system. Compute the $\omega$–limit set of the orbit. What is the $\alpha$–limit set of the orbit?

5. Assume that $b > 0$ and $a = 0$ in the two–dimensional system (2). Sketch the phase portrait of the system. What can you say about the nontrivial orbits? What do you conclude about the solutions of the system?