Solutions to Assignment #10

1. A particle moves in the xy–plane along a path determined by the parametric equations

$$\begin{cases} x = t; \\ y = t^3 - t, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{1}$$

Compute the velocity and acceleration of the particle.

Solution: The parametric equations in (1) define a path $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\sigma(t) = t\hat{\imath} + (t^3 - t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

which locates the particle at any time t. Then, the velocity of the particle is

$$\dot{\sigma}(t) = \hat{\imath} + (3t^2 - 1)\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

and its acceleration is

$$\ddot{\sigma}(t) = 6t\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

□

2. The acceleration of a particle moving in the xy–plane is given $\ddot{\sigma}(t) = -\hat{\jmath}$, for all $t \in \mathbb{R}$. Assume that at time $t = 0$ the velocity of the particle is $\dot{\sigma}(0) = \hat{\imath}$ and the particle is located at the point $(0, 4)$.

   (a) Determine the velocity of the particle at any time $t \geqslant 0$.

   Solution: The vector equation

$$\ddot{\sigma}(t) = -\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

is equivalent to the system of differential equations

$$\begin{cases} \ddot{x} = 0; \\ \ddot{y} = -1, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{2}$$

Integrating the equations in (2) yields

$$\begin{cases} \dot{x} = c_1; \\ \dot{y} = -t + c_2, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{3}$$

where c_1 and c_2 are constants of integration.

The initial condition $\dot{\sigma}(0) = \hat{\imath}$ is equivalent to

$$\dot{x}(0) = 1 \quad \text{and} \quad \dot{y}(0) = 0.$$

Substituting these into the equations in (3) yields

$$c_1 = 1 \quad \text{and} \quad c_2 = 0;$$

so that, in view of (3),

$$\left\{ \begin{array}{rcl} \dot{x} & = & 1; \\ \dot{y} & = & -t, \end{array} \right. \qquad \text{for } t \in \mathbb{R}. \tag{4}$$

Thus, the velocity of the particle is

$$\dot{\sigma}(t) = \hat{\imath} - t\hat{\jmath}, \qquad \text{for } t \geqslant 0.$$

□

(b) Determine the path $\sigma(t)$ of the particle for all time $t \geqslant 0$.
***Solution**:* Integrate the equations in (4) to get

$$\left\{ \begin{array}{rcl} x(t) & = & t + c_1; \\ y(t) & = & -\dfrac{1}{2}t^2 + c_2, \end{array} \right. \qquad \text{for } t \in \mathbb{R}, \tag{5}$$

where c_1 and c_2 are constants of integration.
By virtue of the initial condition

$$x(0) = 0 \quad \text{and} \quad y(0) = 4,$$

we obtain from (5) that

$$c_1 = 0 \quad \text{and} \quad c_2 = 4.$$

Consequently, (5) yields the parametric equations

$$\left\{ \begin{array}{rcl} x & = & t; \\ y & = & -\dfrac{1}{2}t^2 + 4, \end{array} \right. \qquad \text{for } t \in \mathbb{R}, \tag{6}$$

Hence, the path of the particle is

$$\sigma(t) = t\hat{\imath} + \left(-\frac{1}{2}t^2 + 4\right)\hat{\jmath}, \qquad \text{for } t \geqslant 0.$$

□

(c) Sketch the curve traced by the path σ obtained in part (b).

Solution: From the first equation in (6) we get that $t = x$; substituting this value of t in the second equation in (6) then yields

$$y = -\frac{1}{2}x^2 + 4, \tag{7}$$

which is the equation of parabola that opens downwards and with vertex at $(0, 4)$. A sketch of that parabola is shown in Figure 1. □

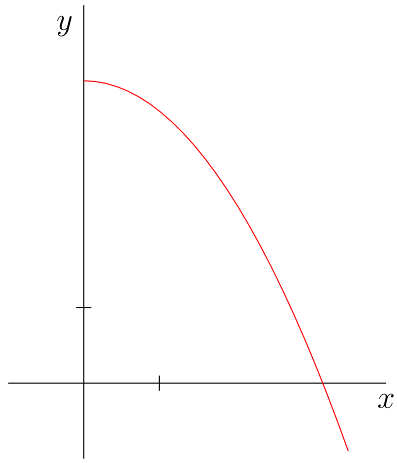

Figure 1: Sketch of path in Problem 2

(d) Determine the time $t > 0$ when the particle is on the x–axis. What are the coordinates of that point?

Solution: The graph of the equation in (7) meets the x–axis when $y = 0$; or, when $x = \pm 2\sqrt{2}$. Thus, according to the first equation in (6, the time $t > 0$ when the particle is on the x–axis is $t = 2\sqrt{2}$. The coordinates of the point of intersection at $(2\sqrt{2}, 0)$. □

3. Assume that acceleration of a particle moving in the plane at any time t is given by $\ddot{\sigma}(t) = \hat{\imath} + 2\hat{\jmath}$, for all $t \in \mathbb{R}$.

   Compute the path σ given that $\sigma(0) = (0, 0)$ and $\sigma'(0) = \hat{\imath} + \hat{\jmath}$.

   Solution: Let

   $$\sigma(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

   where x and y are differentiable functions of t.

We are given that

$$\ddot{\sigma}(t) = \hat{\imath} + 2\hat{\jmath}, \quad \text{for } t \in \mathbb{R}.$$

Consequently,

$$\begin{cases} \ddot{x} = 1; \\ \ddot{y} = 2, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{8}$$

Integrating the equations in (8) yields

$$\begin{cases} \dot{x} = t + c_1; \\ \dot{y} = 2t + c_2, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{9}$$

where c_1 and c_2 are constants of integration.

The initial condition

$$\dot{\sigma}(0) = \hat{\imath} + \hat{\jmath}$$

implies that

$$\dot{x}(0) = 1 \quad \text{and} \quad \dot{y}(0) = 1;$$

so, using the equations in (9),

$$c_1 = 1 \quad \text{and} \quad c_2 = 1.$$

Thus, substituting these into the equations in (9),

$$\begin{cases} \dot{x} = t + 1; \\ \dot{y} = 2t + 2, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{10}$$

Integrate the equations in (10) to get

$$\begin{cases} x = \frac{1}{2}t^2 + t + c_1; \\ y = t^2 + 2t + c_2, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{11}$$

where c_1 and c_2 are constants of integration.

The initial condition $\sigma(0) = (0, 0)$ is equivalent to

$$x(0) = 0 \quad \text{and} \quad y(0) = 0.$$

It then follows from (11) that

$$c_1 = 0 \quad \text{and} \quad c_2 = 0.$$

Substituting these values into (11) then yields

$$\begin{cases} x = \frac{1}{2}t^2 + t; \\ y = t^2 + 2t, \end{cases} \quad \text{for } t \in \mathbb{R}.$$

Consequently,

$$\sigma(t) = \left(\frac{1}{2}t^2 + t\right)\hat{\imath} + (t^2 + 2t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R}.$$

□

4. Use the law of conservation of momentum to determine the path of a particle that is at the point $(0, 1)$ at time $t = 0$ and has velocity $\dot{\sigma}(0) = \hat{\imath} + 2\hat{\jmath}$ at that time, assuming that there no forces act on the particle at any time. Describe and sketch the path.

   ***Solution**:* In this case, the law of conservation of momentum

$$m\ddot{\sigma} = F,$$

   yields

$$\ddot{\sigma} = \mathbf{0}, \tag{12}$$

   since we are assuming that no forces act on the particle at any time t.

   Setting

$$\sigma(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R},$$

   where x and y are differentiable functions of t. the vector equation in (12) is equivalent to the system of differential equations

$$\begin{cases} \ddot{x} = 0; \\ \ddot{y} = 0, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{13}$$

   Integrating the equations in (13) yields

$$\begin{cases} \dot{x} = c_1; \\ \dot{y} = c_2, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{14}$$

   where c_1 and c_2 are constants of integration.

   The initial condition

$$\dot{\sigma}(0) = \hat{\imath} + 2\hat{\jmath}$$

implies that

$$\dot{x}(0) = 1 \quad \text{and} \quad \dot{y}(0) = 2;$$

so, using the equations in (14),

$$c_1 = 1 \quad \text{and} \quad c_2 = 2.$$

Thus, substituting these into the equations in (14),

$$\begin{cases} \dot{x} &= 1; \\ \dot{y} &= 2, \end{cases} \qquad \text{for } t \in \mathbb{R}. \tag{15}$$

Integrate the equations in (15) to get

$$\begin{cases} x &= t + c_1; \\ y &= 2t + c_2, \end{cases} \qquad \text{for } t \in \mathbb{R}, \tag{16}$$

where c_1 and c_2 are constants of integration.

The initial condition $\sigma(0) = (0, 1)$ is equivalent to

$$x(0) = 0 \quad \text{and} \quad y(0) = 1.$$

It then follows from (16) that

$$c_1 = 0 \quad \text{and} \quad c_2 = 1.$$

Substituting these values into (16) then yields

$$\begin{cases} x &= t; \\ y &= 2t + 1, \end{cases} \qquad \text{for } t \in \mathbb{R}.$$

Consequently,

$$\sigma(t) = t\hat{i} + (2t + 1)\hat{j}, \quad \text{for } t \in \mathbb{R},$$

which we can rewrite as

$$\sigma(t) = \hat{j} + t(\hat{i} + 2\hat{j}), \quad \text{for } t \in \mathbb{R},$$

which is the vector–parametric equation of a straight line through the point $(0, 1)$ in the direction of the vector $\dot{\sigma}(0) = \hat{i} + 2\hat{j}$. A sketch of this line is shown in Figure 2. □

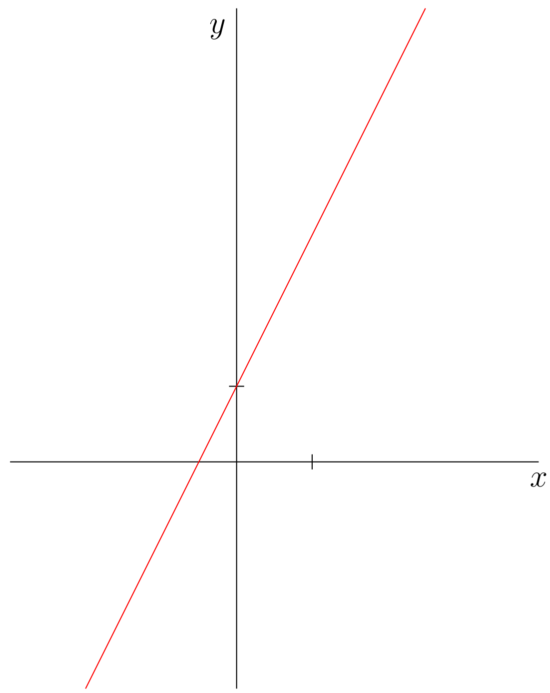

Figure 2: Sketch of path in Problem 4

5. A particle of mass m (in kilograms) is moving along a path in the xy–plane given by $\sigma(t) = R\cos(\omega t)\hat{\imath} + R\sin(\omega t)\hat{\jmath}$, for $t \in \mathbb{R}$, where R is measured in meters and t is measured in seconds.

   (a) Compute the velocity and acceleration of the particle at any time t, and sketch them at a point $\sigma(t)$ on the path.

   ***Solution**:* Given the path $\sigma \colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\sigma(t) = R\cos(\omega t)\hat{\imath} + R\sin(\omega t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R}, \tag{17}$$

   compute

$$\dot{\sigma}(t) = -R\omega\sin(\omega t)\hat{\imath} + R\omega\cos(\omega t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R}, \tag{18}$$

   where we have used the Chain–Rule.

   Similarly, taking the derivative with respect to t to the velocity vector in (18) yields the acceleration vector

$$\ddot{\sigma}(t) = -R\omega^2\cos(\omega t)\hat{\imath} - R\omega^2\sin(\omega t)\hat{\jmath}, \quad \text{for } t \in \mathbb{R}. \tag{19}$$

Note that the acceleration vector in (19) can be written as

$$\ddot{\sigma}(t) = -\omega^2\sigma(t), \qquad \text{for } t \in \mathbb{R} \tag{20}$$

where $\sigma(t)$ is the position vector given in (17).

It follows from (20) that the acceleration vector points in a direction opposite that of the position vector. See Figure 3 for a sketch of $\sigma(t)$, $\dot{\sigma}(t)$

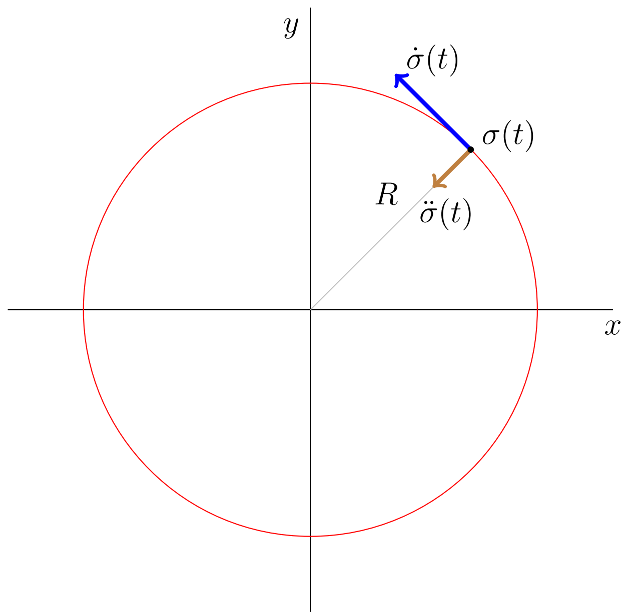

Figure 3: Sketch of path in Problem 5

and $\ddot{\sigma}(t)$, for some time t. Observe that the curve in the xy–plane traced by the path σ is the circle of radius R centered at the origin.

Note that $\dot{\sigma}(t)$ is perpendicular to $\sigma(t)$. This can be seen to be the case by computing the dot product

$$\sigma(t) \cdot \dot{\sigma}(t) = 0,$$

in view of (17) and (18). □

(b) Let $\theta(t)$ denote the angle that $\sigma(t)$ makes with the positive x–axis. Compute $\dot{\theta}$. Give the units of $\dot{\theta}$.

Solution: Using the formula

$$\sigma(t) \cdot \hat{i} = \|\sigma(t)\|\|\hat{i}\| \cos(\theta(t)), \qquad \text{for all } t \in \mathbb{R},$$

we get that

$$R\cos(\theta(t)) = R\cos(\omega t),$$

from which we get that

$$\theta(t) = \omega t + \phi. \tag{21}$$

for some constant ϕ.

Taking the derivative with respect to t on both sides of (21) yields

$$\dot{\theta} = \omega.$$

Consequently, the units of ω are radians per time. □

(c) Use the law of conservation of momentum to compute the magnitude and direction of the force acting on the particle at time t.

***Solution**:* The Law of Conservation of Momentum states that

$$m\ddot{\sigma} = F, \tag{22}$$

where m is the mass of the particle and F is the vector sum of the forces acting on the particle.

Combining (20) and (22) we get that

$$F = -m\omega^2\sigma. \tag{23}$$

So that F is parallel to σ and pointing towards the the origin (the center of the circular path).

It follows from (23) that the magnitude of F is

$$\|F\| = mR\omega^2.$$

□