Solutions to Assignment #14

1. Let A be the 2×2 matrix and suppose that v is a nonzero vector in $\mathbb{R}^2$ such that

$$A\text{v} = \lambda\text{v}, \tag{1}$$

for some scalar λ.

Define the path $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ce^{\lambda t}\text{v}, \text{ for all } t \in \mathbb{R}, \tag{2}$$

where c is scalar constant. Verify that $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of the system of first order differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}$$

where the dot above the variable name indicates derivative with respect to t.

Suggestion: Differentiate on both sides of (2) with respect to t and use (1).

Notation. The function in (2) is called a line solution of the system in (3).

***Solution**:* Take the derivative with respect to t on both sides of the equation in (2) to get

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} &= \frac{d}{dt}\left(ce^{\lambda t}\text{v}\right) \\ &= c\lambda e^{\lambda t}\text{v} \\ &= ce^{\lambda t}(\lambda\text{v}); \end{aligned}$$

so that, in view of (1),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = ce^{\lambda t}A\text{v}.$$

Consequently, using the properties of matrix multiplication,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A\left(ce^{\lambda t}\text{v}\right).$$

Thus, in view of the definition of $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in (2),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

which shows that the function $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ defined in (2) solves the system of first–order differential equations given in (3). □

2. Let A denote the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $\mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Verify that $A\mathrm{v}_1 = \lambda_1 \mathrm{v}_1$, where $\lambda_1 = -1$; and $A\mathrm{v}_2 = \lambda_2 \mathrm{v}_2$, where $\lambda_2 = 1$.

***Solution**:* Compute

$$\begin{aligned} A\mathrm{v}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= -\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

which shows that $A\mathrm{v}_1 = \lambda_1 \mathrm{v}_1$, where $\lambda_1 = -1$.

Similarly,

$$\begin{aligned} A\mathrm{v}_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

which shows that $A\mathrm{v}_2 = \lambda_2 \mathrm{v}_2$, where $\lambda_2 = 1$. □

3. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = y; \\[2ex] \dfrac{dy}{dt} = x. \end{cases} \tag{4}$$

   (a) Show that the system in (4) can be written in vector form as in (3) where A is the matrix given in Problem 2.

   ***Solution**:* Write the system (4) in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$

   which can be written in terms of matrix multiplication as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

   Thus, according to (5), the system in (4) is of the form as (3), where A is the matrix given in Problem 3. □

   (b) Let v_1 and v_2 be the vectors given in Problem 3, $\lambda_1 = -1$ and $\lambda_2 = 1$. Use the result in Problem 1 to show that

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = e^{\lambda_1 t}\mathrm{v}_1 \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = e^{\lambda_2 t}\mathrm{v}_2, \quad \text{for all } t \in \mathbb{R}, \tag{6}$$

   define solutions of the system in (4).

   ***Solution**:* Apply the result of Problem 1 to each of the vector–valued functions defined in (6) to get that

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \end{pmatrix} = A \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \text{ and } \begin{pmatrix} \dot{x}_2(t) \\ \dot{y}_2(t) \end{pmatrix} = A \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \text{ for all } t \in \mathbb{R}, \tag{7}$$

   which shows that the functions defined in (6) solve the system in (4). □

4. Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be the paths defined in Problem 3.

   Verify that the function $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}, \tag{8}$$

solves the system in (4).

Solution: Take the derivative with respect to t on both sides of the expression in (8) to compute

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c_1 \begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \end{pmatrix} + c_2 \begin{pmatrix} \dot{x}_2(t) \\ \dot{y}_2(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R};$$

so that, using (7),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c_1 A \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 A \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R};$$

thus, applying the distributive property of matrix multiplication,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \left(c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \right), \quad \text{for } t \in \mathbb{R}. \tag{9}$$

Comparing (9) and (8), we see that

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

which shows that the function $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ defined in (8) solves the system in (4). □

5. Use the function given in (8) to sketch the flow of the vector field

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{10}$$

Solution: Since the vector field in (10) is given by

$$F \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \tag{11}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It then follows from the results in Problem 2, Problem 3 and Problem 4, that the general solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{12}$$

is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \mathrm{v}_1 + c_2 e^{\lambda_2 t} \mathrm{v}_2, \quad \text{for all } t \in \mathbb{R}. \tag{13}$$

where

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{14}$$

and the scalars λ_1 and λ_2 are given by

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 1.$$

We can then rewrite (13) as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \mathrm{v}_1 + c_2 e^{t} \mathrm{v}_2, \quad \text{for all } t \in \mathbb{R}, \tag{15}$$

where the vectors v_1 and v_2 are given in (14).

We can use the expression in (15) for the solutions of the system in (12) to sketch the flow of the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given in (10).

By taking $c_1 = c_2 = 0$ in (15), we obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

This is the equilibrium solution at the origin sketched as a dot in Figure 1.

The case $c_1 = 0$ and $c_2 \neq 0$ yields the solutions

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_2 e^{t} \mathrm{v}_2, \quad \text{for all } t \in \mathbb{R}, \tag{16}$$

where $c_2 \neq 0$.

The equation in (16) is the vector–parametric equation of a half–line emanating from the origin in the direction of the vector $\mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, in the case $c_2 > 0$, since $e^t > 0$ for all $t \in \mathbb{R}$. For the case $c_2 < 0$, the half–line parametrized by (16) is in the direction opposite to that of v_2. Both trajectories parametrized by (16) point away from the origin since e^t increases as t increases. The directions along these trajectories are indicated by arrows on the half–lines shown in Figure 1.

For the case $c_1 \neq 0$ and $c_2 = 0$, we obtain from (15) the vector–parametric equation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \mathrm{v}_1, \tag{17}$$

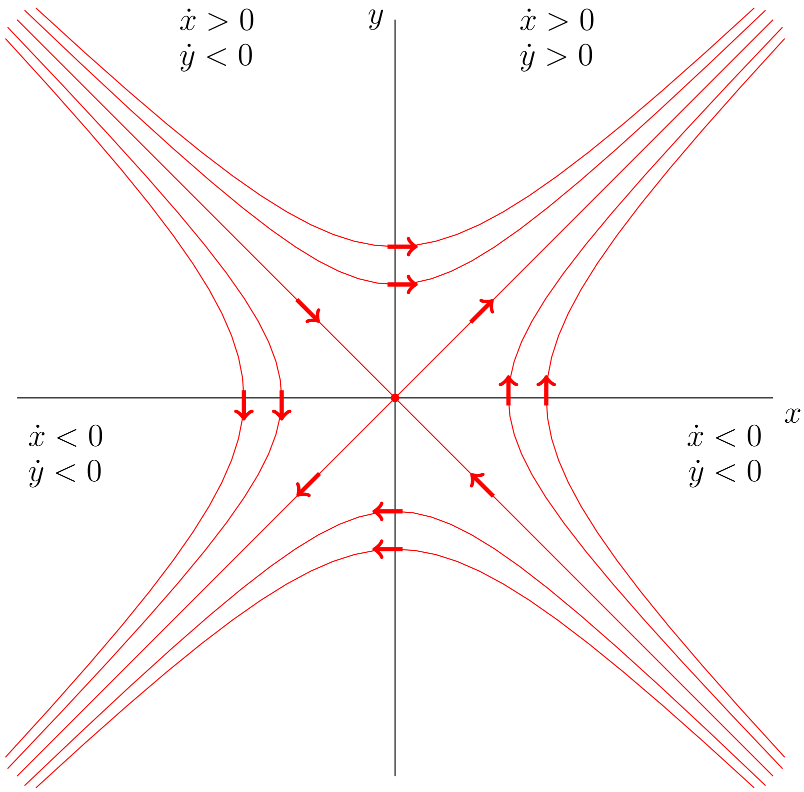

Figure 1: Sketch of Flow of Vector Field

where $c_1 \neq 0$.

The equation in (17) parametrizes half–lines in the direction of $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, for the case $c_1 > 0$, and in the opposite direction in the case $c_1 < 0$. Both trajectories tend towards the origin because e^{-t} decreases to 0 as t increases. These are sketched in Figure 1.

To sketch the trajectories parametrized by (15) for the case $c_1 \neq 0$ and $c_2 \neq 0$, use the directions prescribed by the signs of $\dot{x}$ and $\dot{y}$ given by the differential equations in the system in (4). These directions are shown in the sketch in Figure 1. $\square$