Solutions to Assignment #16

In problems (1)–(5), for the given the two–dimensional, linear system, (a) compute and sketch line–solutions, if any; (b) sketch the nullclines; (c) sketch the phase portrait; and (d) describe the nature of the stability, or unstability, of the origin.

1. $\begin{cases} \dot{x} &= -2y; \\ \dot{y} &= x - 3y. \end{cases}$

Solution:

(a) Write the system in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}.$$

The characteristic polynomial of the matrix A is

$$p_{_A}(\lambda) = \lambda^2 + 3\lambda + 2,$$

which factors into

$$p_{_A}(\lambda) = (\lambda + 2)(\lambda + 1).$$

Thus, the matrix A has two distinct, real eigenvalues:

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -1.$$

Next, we find eigenvectors corresponding to the eigenvalues λ_1 and λ_2. To find an eigenvector corresponding to $\lambda_1 = -2$, compute nontrivial solutions of the system

$$(A - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where I is the 2×2 identity matrix, or

$$\begin{cases} (0 - (-2))x - 2y &= 0; \\ x + (-3 - (-2))y &= 0, \end{cases}$$

or

$$\left\{\begin{array}{rcl} 2x-2y & = & 0; \\ x-y & = & 0, \end{array}\right.$$

which reduces to the equation

$$x - y = 0. \tag{1}$$

To find solutions of the equation in (1), solve for x,

$$x = y,$$

and set $y = t$, where t is a real parameter. Then, the solutions of (1) are the vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{2}$$

Taking $t = 1$ in (2) yields the vector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3}$$

which is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1 = -2$.

To find an eigenvector corresponding to $\lambda_1 = -1$, compute nontrivial solutions of the system

$$(A - \lambda_2 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\left\{\begin{array}{rcl} (0-(-1))x - 2y & = & 0; \\ x + (-3-(-1))y & = & 0, \end{array}\right.$$

or

$$\left\{\begin{array}{rcl} x-2y & = & 0; \\ x-2y & = & 0, \end{array}\right.$$

which reduces to the equation

$$x - 2y = 0. \tag{4}$$

To find solutions of the equation in (4), solve for x,

$$x = 2y,$$

and set $y = t$, where t is a real parameter. Then, the solutions of (4) are the vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{5}$$

Taking $t = 1$ in (5) yields the vector

$$\mathrm{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{6}$$

which is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_2 = -1$.

Thus, the line solutions of the system in this problem are

$$c_1 e^{-2t}\mathrm{v}_1 \quad \text{and} \quad c_2 e^{-t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are non–zero constants, and v_1 and v_2 are given in (3) and (6), respectively. These are sketched in Figure 1 along with the equilibrium solution $(0, 0)$. Note that the direction of these trajectories along the line four line–solutions in the figures is towards the origin because e^{-2t} and e^{-t} decrease to 0 as t increases.

(b) The $\dot{x} = 0$–nullcline is the line $y = 0$ or the x–axis. On this line, the vector field associated with the system in this problem,

$$F(x, y) = \begin{pmatrix} -2y \\ x - 3y \end{pmatrix}, \tag{7}$$

is vertical. This is indicated by the vertical arrows drawn across the x–axis in Figure 1. Note that the arrows point up for positive values of x, since the field F in (7) points up for $y = 0$ and $x > 0$. By the same token, the arrows point down for negative values of x.

The $\dot{y} = 0$–nullcline of the system in this problem is the line $x - 3y = 0$,

$$y = \frac{1}{3}x.$$

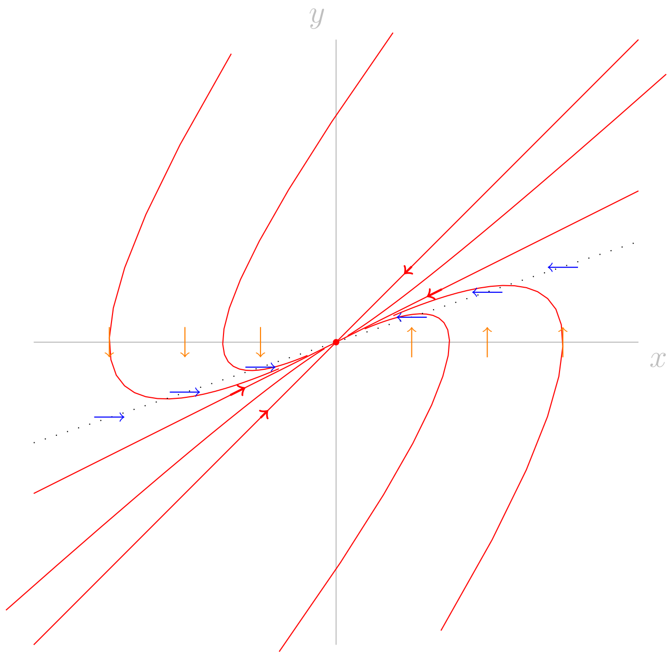

Figure 1: Sketch of phase portrait of system in Problem 1

This line is sketched as a dotted line in the sketch in Figure 1.

On the $\dot{y} = 0$–nullcline, the vector field is horizontal. This is indicated by the horizontal arrows on the nullcline shown in the figure. Note that, according to the definition of the field F in (7), the arrows point to the left above the x–axis ($y > 0$), and point to the right below the x–axis ($y < 0$).

(c) To sketch the phase portrait of the system in this problem we use arrows on the nullclines as guide, as well as the signs of $\dot{x}$ and $\dot{y}$ determined by the differential equations, to sketch a few solution curves. Some of these trajectories are shoen in Figure 1.

(d) Since the eigenvalues of the matrix for this problem are both negative, the origin is an asymptotically stable equilibrium point; it is a sink.

$\square$

2. $$\begin{cases} \dot{x} &= -y; \\ \dot{y} &= x + 2y. \end{cases}$$

Solution:

(a) Write the system in matrix form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}. \tag{8}$$

The characteristic polynomial of the matrix A is

$$p_{_A}(\lambda) = \lambda^2 - 2\lambda + 1,$$

which factors into

$$p_{_A}(\lambda) = (\lambda - 1)^2.$$

Consequently, the matrix A in (8) has only one eigenvalue,

$$\lambda = 1.$$

Next, we find eigenvectors corresponding to $\lambda = 1$ by computing nontrivial solutions of the system

$$(A - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where I is the 2×2 identity matrix, or

$$\left\{ \begin{array}{rcl} (0-1))x - y & = & 0; \\ x + (2-1))y & = & 0, \end{array} \right.$$

or

$$\left\{ \begin{array}{rcl} -x - y & = & 0; \\ x + y & = & 0, \end{array} \right.$$

which reduces to the equation

$$x + y = 0. \tag{9}$$

To compute the solutions of (9), solve for x,

$$x = -y,$$

and set $y = -t$ to get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{10}$$

Taking $t = 1$ in (10) yields the vector

$$\mathrm{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{11}$$

which is an eigenvector of the matrix A in (8) corresponding to the eigenvalue $\lambda = 1$. Hence, the line solutions of the system in this problem are given by

$$ce^t\mathrm{v}, \qquad \text{for } t \in \mathbb{R}, \tag{12}$$

for $c \neq 0$. These solutions, along with the equilibrium solution $(0, 0)$, are sketched in Figure 2. Note that the trajectories along the line–solutions in

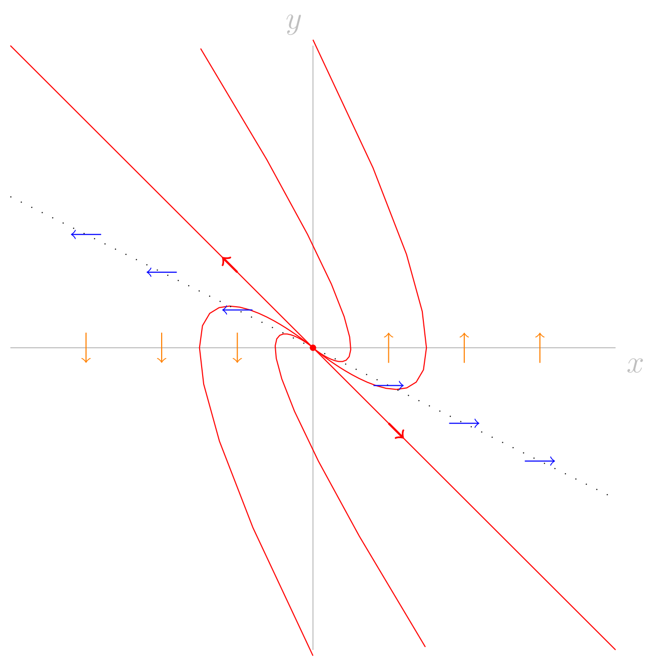

Figure 2: Sketch of phase portrait of system in Problem 2

(12) tend away from the origin because e^t increases as t increases.

(b) The $\dot{x} = 0$–nullcline is the line $y = 0$, or the x–axis. On this line, the direction of the vector field

$$F(x,y) = \begin{pmatrix} -y \\ x+2y \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2, \tag{13}$$

is vertical. This is indicated by the vertical arrows sketched along the x–axis in the sketch in Figure 2. Note that the arrows point up for $x > 0$ and down for $x < 0$.

The $\dot{y} = 0$–nullcline is the line $x + 2y = 0$, or

$$y = -\frac{1}{2}x. \tag{14}$$

On this line, the vector field in (13) is horizontal.

The line in (14) is sketched in Figure 2 as a doted line with horizontal arrows across it. The arrows point to the right below the x-axis (for $y < 0$) and to the left above the x–axis (for $y > 0$).

(c) To sketch the phase portrait of the system in this problem, we use the nullclines sketched in Figure 2, as well as the direction vectors along the nullclines, to sketch a few possible trajectories of the system; a few of these are sketched in Figure 2.

(d) Since all the trajectories, other than the equilibrium solution, turn away from the origin, the equilibrium point $(0,0)$ is an unstable node.

□

3. $\begin{cases} \dot{x} &= -x + 4y; \\ \dot{y} &= -2x + 3y. \end{cases}$

Solution:

(a) Write the system in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}. \tag{15}$$

The characteristic polynomial of the matrix A in (15) is

$$p_A(\lambda) = \lambda^2 - 2\lambda + 5,$$

which we can rewrite as

$$p_A(\lambda) = \lambda^2 - 2\lambda + 1 + 4,$$

or

$$p_A(\lambda) = (\lambda - 1)^2 + 4. \tag{16}$$

It follows from (16) that the equation

$$p_A(\lambda) = 0$$

has no real solutions. Hence, the system in this problem does not have line solutions.

(b) The $\dot{x} = 0$–nullcline of the system in this problem is the line

$$-x + 4y = 0,$$

or

$$y = \frac{1}{4}x. \tag{17}$$

This line is sketched in Figure 3 with the verticals arrows across it indicating the directions of the vector field

$$F(x, y) = \begin{pmatrix} -x + 4y \\ -2x + 3y \end{pmatrix}. \tag{18}$$

The $\dot{y} = 0$–nullcline is the line $-2x + 3y = 0$, or

$$y = \frac{2}{3}x. \tag{19}$$

This line is sketched in Figure 3 along with the horizontal arrows indicating the direction of the field F in (18).

The directions along the arrows on the nullclines can be determined by looking at the signs of $\dot{x}$ and $\dot{y}$. These are given by the differential equations of the system in this problem and are displayed in the sketch in Figure 3.

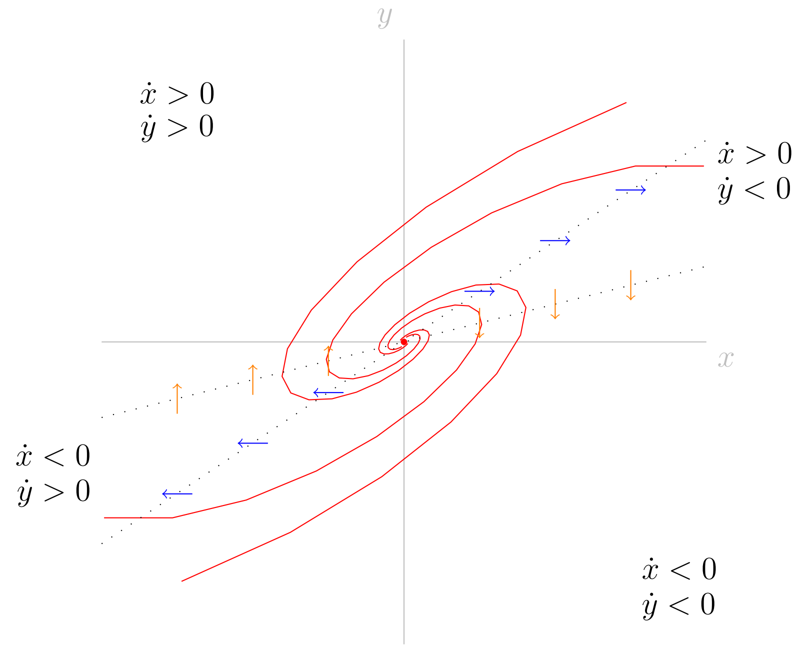

Figure 3: Sketch of phase portrait of system in Problem 3

(c) The solutions of the equation $p_{_A}(\lambda) = 0$, where $p_{_A}(\lambda)$ is given in (16), are the complex numbers

$$\lambda = 1 \pm 2i;$$

thus, the eigenvalues of the matrix A in (15) are complex with positive real part. Hence, the trajectories of the system in this problem will spiral away from the origin in the clockwise direction, as indicated by the arrows on the nullclines in the sketch in Figure 3. A few of these trajectories are sketched in the figure.

(d) The origin is unstable; it is a spiral source.

□

4. $\left\{ \begin{array}{rcl} \dot{x} & = & y; \\ \dot{y} & = & -4x. \end{array} \right.$

***Solution**:*

(a) Write the system in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}. \tag{20}$$

The characteristic polynomial of the matrix A in (20) is

$$p_{_A}(\lambda) = \lambda^2 + 4, \tag{21}$$

which has no real roots. Hence, the matrix A in (20) has no real eigenvalues. Consequently, the system in in this problem has not line–solutions.

(b) The $\dot{x} = 0$–nullcline is the line $y = 0$, or the x–axis. This line is sketched in Figure 4, along with vertical line segments across it indicating the direction

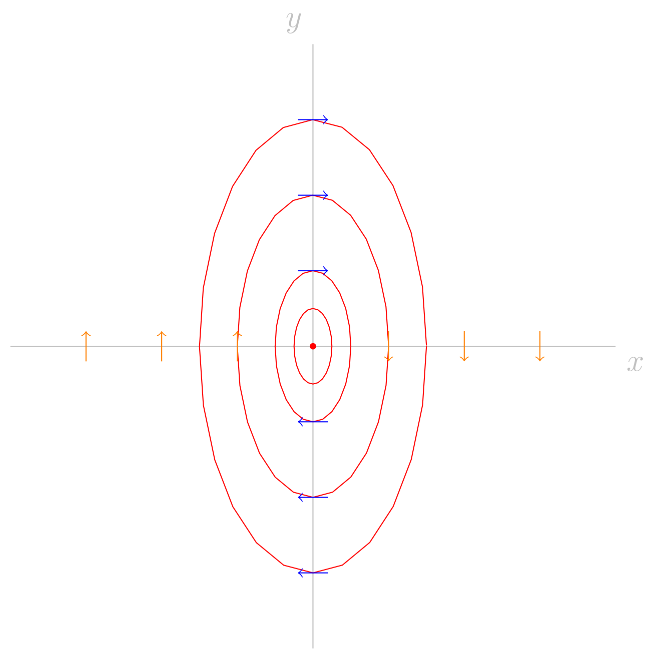

Figure 4: Sketch of phase portrait of system in Problem 4

of the vector field

$$F(x, y) = \begin{pmatrix} y \\ -4x \end{pmatrix}. \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{22}$$

along the nullcline. Note that the arrows point down for $x > 0$, and up for $x < 0$.

The $\dot{y} = 0$–nullcline is the line $x = 0$ (the y–axis). On this line the vector field in (22) is horizontal. This is indicated in Figure 4 by horizontal arrows. The arrows point to the right for above the x–axis ($y > 0$), to the left below the x–axis ($y < 0$).

(c) The roots of the characteristic polynomial $p_{_A}(\lambda)$ in 21) are the complex numbers $\lambda = \pm 2i$. Thus, the eigenvalues of the matrix A in (20) are the purely imaginary numbers $\pm 2i$. Since, the real part of the eigenvalues is 0, and the system in this problem is linear, the trajectories of the system in this problem are closed curves around the origin. Indeed, the trajectories are ellipses centered at the origin given by the equation

$$x^2 + \frac{y^2}{4} = c, \tag{23}$$

where c is a non–negative constant. These curves have a clockwise orientation according to the directions of the field in (22) along the nullclines. Note that the solution curves given by the equations in (23) also include the equilibrium solution, $(0, 0)$, for the case $c = 0$.

(d) The equilibrium point, $(0, 0)$, of the system in this problem is neutrally stable; the origin is a center.

□

5. $\begin{cases} \dot{x} &= -3x + 5y; \\ \dot{y} &= -2x + 3y. \end{cases}$

Solution:

(a) Write the system in vector form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}. \tag{24}$$

The characteristic polynomial of the matrix A in (24) is

$$p_{_A}(\lambda) = \lambda^2 + 1, \tag{25}$$

which has no real roots. Hence, the matrix A in (24) has no real eigenvalues. Consequently, the system in in this problem has not line–solutions.

(b) The $\dot{x} = 0$–nullcline is the line $-3x + 5y = 0$, or

$$y = \frac{3}{5}x. \tag{26}$$

This line is sketched in Figure 5.

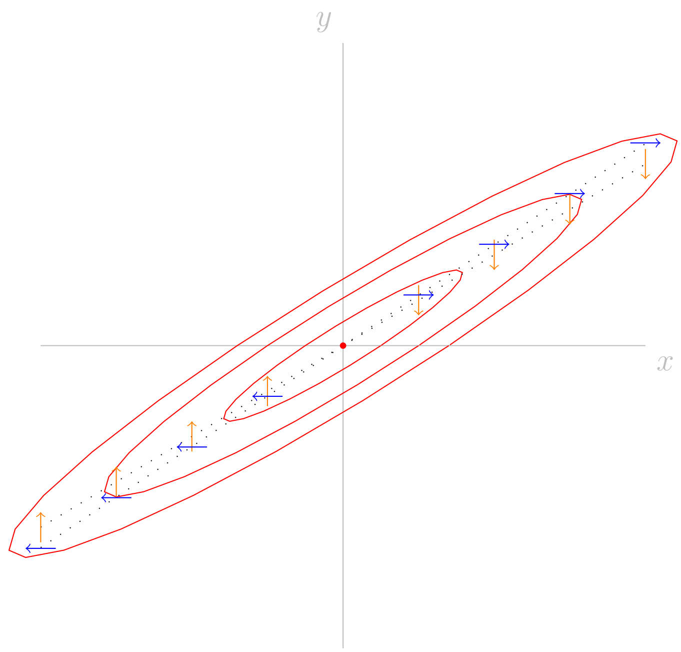

Figure 5: Sketch of phase portrait of system in Problem 5

On the line in (26), the vector field

$$F(x,y) = \begin{pmatrix} -3x + 5y \\ -2x + 3y \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{27}$$

is vertical. This is indicated by the vertical arrows on in the sketch in Figure 5. The arrows point down for positive values of x and up for negative values. To see why this is the case, observe that, on the line in (26), $-3x + 5y = 0$; so, we can replace y by $\dfrac{3}{5}x$ in the definition of the field F in (27) to get

$$F(x,y) = \begin{pmatrix} 0 \\ -0.2x \end{pmatrix}, \quad \text{for } y = \frac{3}{5}x.$$

The $\dot{y} = 0$–nullcline is the line $-2x + 3y = 0$, or

$$y = \frac{2}{3}x. \tag{28}$$

This line is sketched in Figure 5 along with the horizontal arrows indicating the direction of the field F in (27). Note that, on the line in (28), the field is given by

$$F(x, y) = \begin{pmatrix} y/3 \\ 0 \end{pmatrix}, \quad \text{for } y = \frac{2}{3}x.$$

Thus, the arrows on the line in (28) point to the right above the x–axis ($y > 0$), and to the left below the x–axis ($y < 0$).

(c) The roots of the characteristic polynomial $p_A(\lambda)$ in 25) are the complex numbers $\lambda = \pm i$. Thus, the eigenvalues of the matrix A in (24) are the purely imaginary numbers $\pm i$. Since, the real part of the eigenvalues is 0, and the system in this problem is linear, the trajectories of the system in this problem are ellipses centered at the origin and oriented in the clockwise sense. A few of these ellipses are sketched in Figure 5.

(d) The equilibrium point, $(0, 0)$, of the system in this problem is neutrally stable; the origin is a center.

□