Solutions to Assignment #17

1. The expression $f(x,y) = 2 - \sqrt{4 - x^2 - y^2}$ defines a function of two variables

   (a) Give the domain of f.

   ***Solution**:* The value $f(x,y)$ is real provided that $4 - x^2 - y^2 \geqslant 0$; thus, $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 4\}$, the disc of radius 2 around the origin in $\mathbb{R}^2$. □

   (b) Sketch a few of the contour curves: $f(x,y) = c$; indicate values of values c for which contour curves exist.

   ***Solution**:* The contour curves are the graphs of the equations

   $$2 - \sqrt{4 - x^2 - y^2} = c, \qquad \text{for } 0 \leqslant c \leqslant 2,$$

   or

   $$x^2 + y^2 = 4 - (2 - c)^2, \qquad \text{for } 0 \leqslant c \leqslant 2.$$

   Thus the contour curves are concentric circles around the origin of radius ranging from 0 to 2 (these include the origin). A sketch of the contour plot is shown in Figure 1. □

   (c) Sketch the graph of f.

   ***Solution**:* The graph of f is the graph of the equation

   $$z = 2 - \sqrt{4 - x^2 - y^2}, \qquad \text{for } x^2 + y^2 \leqslant 4,$$

   which can be rewritten as

   $$\sqrt{4 - x^2 - y^2} = 2 - z,$$

   or, squaring on both sides

   $$4 - x^2 - y^2 = (2 - z)^2,$$

   or

   $$x^2 + y^2 + (z - 2)^2 = 4. \tag{1}$$

   Thus, according to (1), points on the graph of f are at a fixed distance of 2 from the point $(0,0,2)$; that is, the graph of f consists of points on a sphere of radius 2 centered at the point $(0,0,2)$ in three–dimensional space. Since the values of z are restricted to be between 0 and 2, the graph of f is the lower hemisphere of the sphere pictured in Figure 2. □

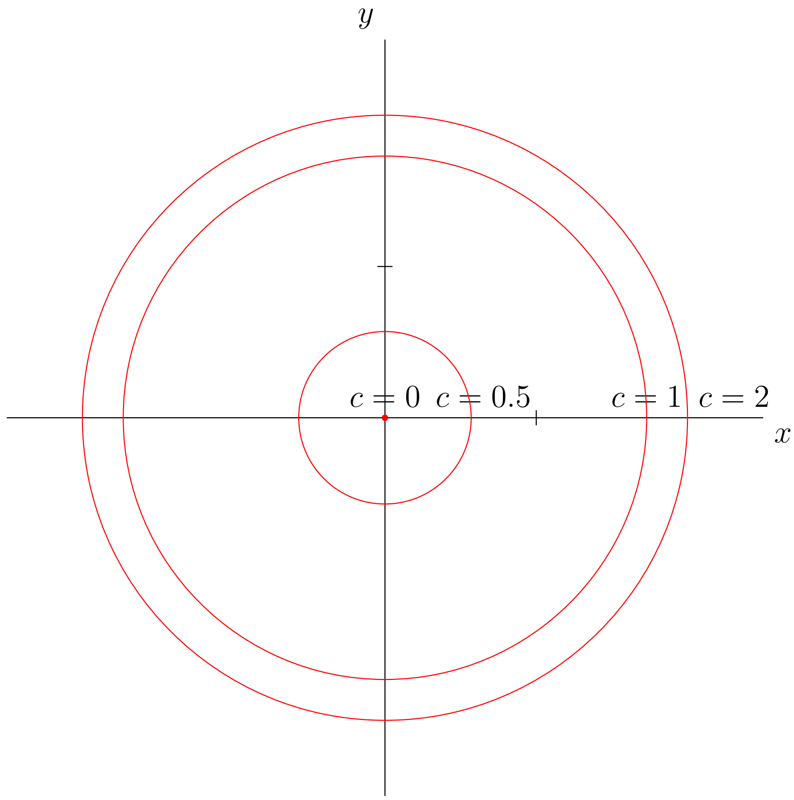

Figure 1: Contour plot for function in Problem 1

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 4 - x^2 - y^2$, for all $(x,y) \in \mathbb{R}^2$.

   (a) Give the domain of f.

   ***Solution**:* There is no restriction on the input values that f can take in; thus, $\text{Dom}(f) = \mathbb{R}^2$. □

   (b) Sketch a few of the contour curves of the graph of f.

   ***Solution**:* The contour curves of the graph of f are graphs of the equations

$$4 - x^2 - y^2 = c,$$

or

$$x^2 + y^2 = 4 - c, \tag{2}$$

for $c \leqslant 4$, in the xy–plane. The graphs of the equations in (2) are concentric circles of radius $\sqrt{4-c}$, for $c \leqslant 4$. Some of these circles are shown in Figure 3. □

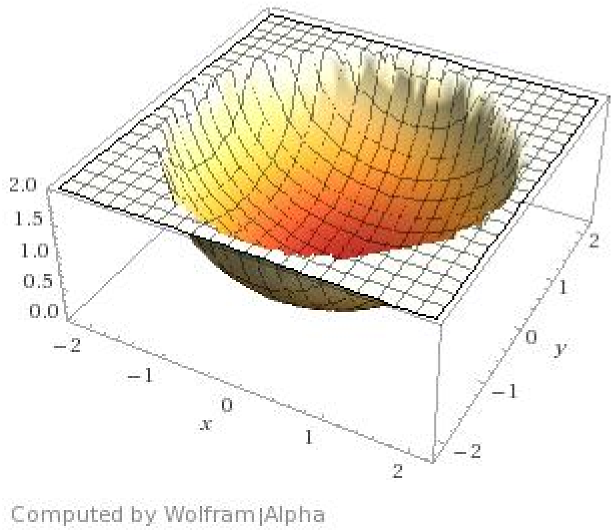

Figure 2: Sketch of graph of $z = 2 - \sqrt{4 - x^2 - y^2}$

(c) Sketch the graph of $z = f(x, y)$.
***Solution**:* Note that the intersection of the graph of $z = f(x, y)$ with the yz–pane is a the parabola given by the graph of the equation

$$z = 4 - y^2,$$

while the intersection with the xz–plane is the parabola given by

$$z = 4 - x^2.$$

The graph of $z = f(x, y)$ is obtained by rotating the graph of any of the parabolic sections about the z–axis. Thus, the graph of $z = f(x, y)$ is a paraboloid of revolution as sketched in Figure 4. □

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 4 - 3x - 2y$, for all $(x, y) \in \mathbb{R}^2$.

(a) Give the domain of f.
***Answer**:* $\text{Dom}(f) = \mathbb{R}^2$. □

(b) Sketch a few of the contour curves of the graph of f.
***Solution**:* The contour curves of the graph of f are straight lines given by

$$4 - 3x - 2y = c,$$

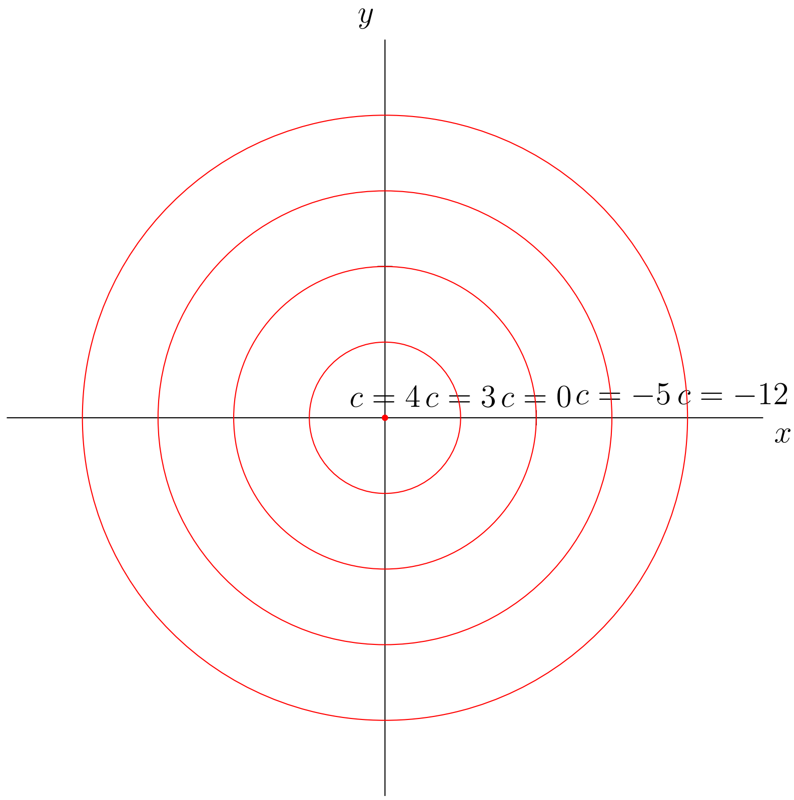

Figure 3: Contour plot for function in Problem 2

or

$$3x + 2y = 4 - c, \tag{3}$$

for $c \in \mathbb{R}$. The lines given in (3) are parallel to each other and have slope $-3/2$. Some of these lines are shown in the contour plot in Figure 5 obtained by using Wolfram Alpha. □

(c) Sketch the graph of $z = f(x, y)$.

***Solution**:* The graph of

$$z = 4 - 3x - 2y$$

is a plane in $\mathbb{R}^3$ with intercepts $(4/3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 4)$. A sketch of the graph of this plane is shown in Figure 6. □

4. Suppose that f is a linear function of x and y that has slope 2 in the x direction and slope 3 in the y–direction.

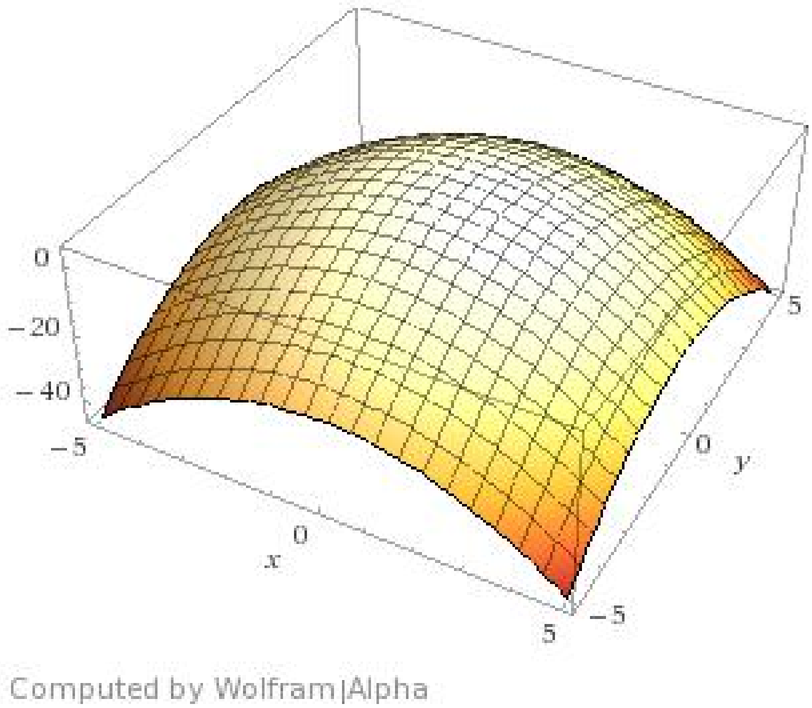

Figure 4: Sketch of graph of $z = 4 - x^2 - y^2$

(a) Determine the change in $z = f(x, y)$ that a change of 0.5 in x and a change of -0.4 in y produces.

***Solution**:* The function f is given by

$$f(x, y) = z_o + 2(x - x_o) + 3(y - y_o), \quad \text{for } (x, y) \in \mathbb{R}^2,$$

for some real values x_o, y_o and z_o; or, by

$$f(x, y) = d + 2x + 3y, \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{4}$$

for some value d.

Using the formula in (4), we compute

$$f(x_1, y_1) = d + 2x_1 + 3y_1, \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{5}$$

and

$$f(x_2, y_2) = d + 2x_2 + 3y_2, \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{6}$$

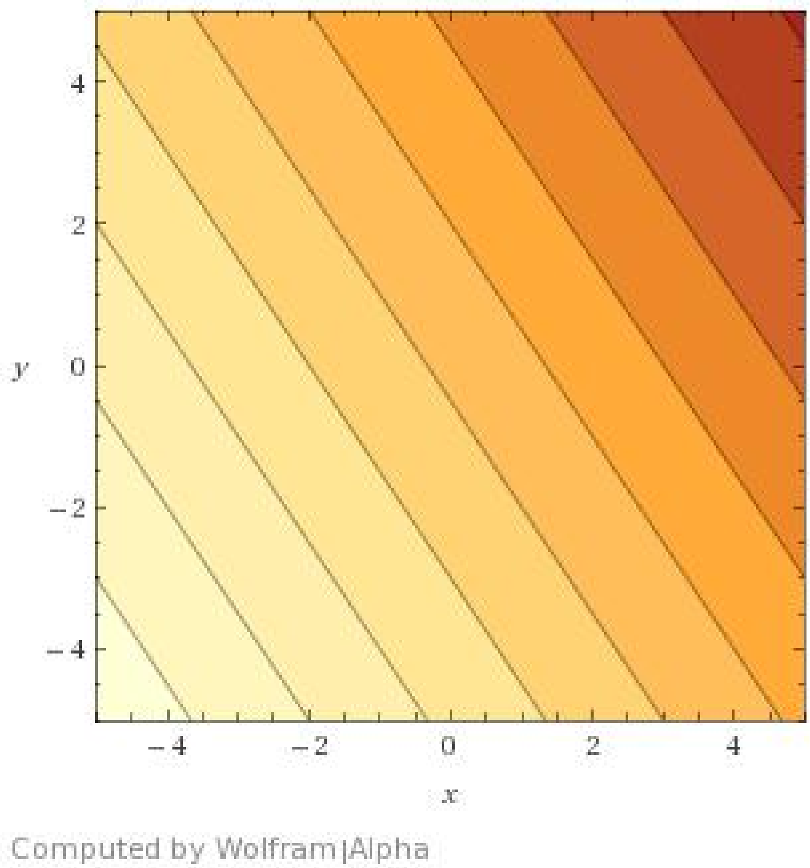

Figure 5: Sketch of Contour Plot in Problem 3

for points (x_1, y_1) and (x_2, y_2), respectively. Setting $z_1 = f(x_1, y_1)$ and $z_2 = f(x_2, y_2)$, and $\Delta z = z_2 - z_1$, the change is z, we get that, by subtracting (5) from (6),

$$\Delta z = 2(x_2 - x_1) + 3(y_2 - y_1),$$

or

$$\Delta z = 2\Delta x + 3\Delta y, \tag{7}$$

where $\Delta x = x_2 - x_1$ is the change in x, and $\Delta y = y_2 - y_1$ is the change in y.

Using (7) with $\Delta x = 0.5$ and $\Delta y = -0.4$, we get that

$$\Delta z = 2(0.5) + 3(-0.4) = -0.2;$$

so that, the change in z is -0.2. □

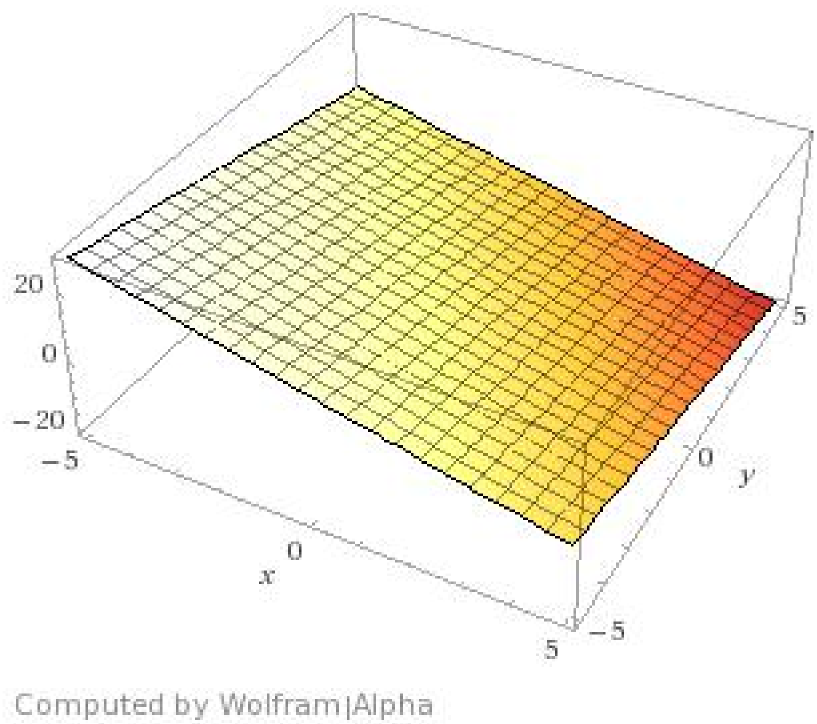

Figure 6: Sketch of graph of $z = 4 - 2x - 3y$

(b) If $f(5,7) = 2$, determine the value of $z = f(x,y)$ when $x = 4.9$ and $y = 7.2$.
Solution: Use $f(5,7) = 2$, together with (4), to determine d in (4) as follows: Solve the equation

$$d + 2(5) + 3(7) = 2$$

for d to get

$$d = -29.$$

Then,

$$f(x,y) = -29 + 2x + 3y, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{8}$$

We can then use (8) to evaluate

$$f(4.9, 7.2) = -29 + 2(4.9) + 3(7.2) = 2.4$$

Alternatively, since the change in x from $(5,7)$ to $(4.9, 7.2)$ is -0.1, and the change in y is 0.2,

$$f(4.9, 7.2) = 2 + 2(-0.1) + 3(0.2) = 2.4.$$

□

5. The graph of a linear function f is a plane passing through the point $(4, 3, -2)$ in three–dimensional space $\mathbb{R}^3$, and having slope 5 in the x–direction and slope -3 in the y–direction.

   (a) Determine a formula for computing $f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

   Solution: Use the formula

   $$f(x, y) = z_o + a(x - x_o) + b(y - y_o), \quad \text{for } (x, y) \in \mathbb{R}^2,$$

   where $a = 5$, $b = -3$, and $(x_o, y_o, z_o) = (4, 3, -2)$, to get

   $$f(x, y) = -2 + 5(x - 4) + (-3)(y - 3), \quad \text{for } (x, y) \in \mathbb{R}^2,$$

   or

   $$f(x, y) = -13 + 5x - 3y, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{9}$$

   □

   (b) Sketch contour lines for the function f.

   Solution: The contour curves of the function $f : \mathbb{R}^2 \to \mathbb{R}$ are the straight lines given by

   $$-13 + 5x - 3y = c,$$

   where $c \in \mathbb{R}$, or

   $$y = \frac{5}{3}x - \frac{c + 13}{3};$$

   that is, parallel lines of slope $5/3$. Some of these contour lines are shown in Figure 7 obtained by using Wolfram Alpha. □

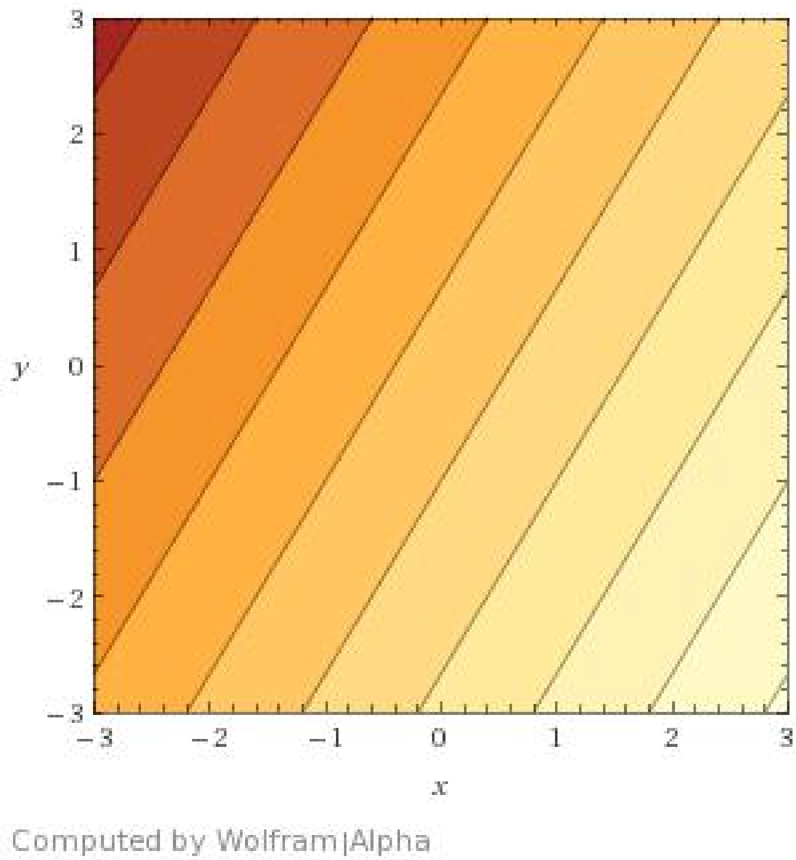

Figure 7: Sketch of Contour Plot in Problem 5