Solutions to Assignment #18

1. Give the formula for computing an affine function, $f\colon \mathbb{R}^2 \to \mathbb{R}$, whose graph is the plane passing through the points $(0,0,0)$, $(0,2,-1)$ and $(-3,0,4)$.

   Sketch the plane.

   ***Solution**:* The expression for f is given by

   $$f(x,y) = d + ax + by, \qquad \text{for } (x,y)\in\mathbb{R}^2, \tag{1}$$

   where a, b and d are determined by solving the equations

   $$\left\{\begin{array}{rcl} f(0,0) & = & 0; \\ f(0,2) & = & -1; \\ f(-3,0) & = & 4, \end{array}\right. \tag{2}$$

   simultaneously.

   It follows from (1) and the first equation in (2) that $d = 0$; thus, we can rewrite (1) as

   $$f(x,y) = ax + by, \qquad \text{for } (x,y)\in\mathbb{R}^2. \tag{3}$$

   Next, use (3) to rewrite the last two equations in (2) as

   $$\left\{\begin{array}{rcl} 2b & = & -1; \\ -3a & = & 4, \end{array}\right.$$

   which can be solved to yield that

   $$b = -\frac{1}{2} \qquad \text{and} \qquad a = -\frac{4}{3}. \tag{4}$$

   Combining (3) and (4) yields

   $$f(x,y) = -\frac{4}{3}x - \frac{1}{2}y, \qquad \text{for } (x,y)\in\mathbb{R}^2. \tag{5}$$

   A sketch of the graph of f given in (5) is shown in Figure 1. $\square$

2. Give the equation for the plane containing the line in the xy–plane where $y = 1$, and the line in the xz–plane where $z = 2$.

   Sketch the plane.

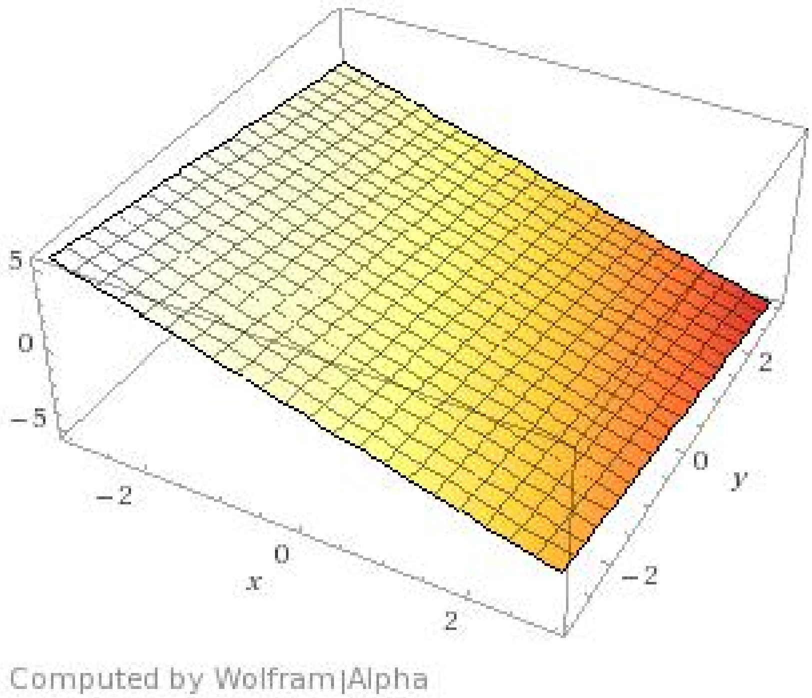

Figure 1: Sketch of Plane in Problem 1

***Solution**:* The plane is the graph of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = d + ax + by, \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{6}$$

where a, b and d are to be determined.

Since the section of the plane in the xy–plane,

$$d + ax + by = 0, \tag{7}$$

is the line

$$y = 1, \tag{8}$$

it follows, by comparing (7) and (8) that

$$a = 0 \quad \text{and} \quad -\frac{d}{b} = 1. \tag{9}$$

On the other hand, since the section of the plane in the xz–plane,

$$z = d + ax, \tag{10}$$

is

$$z = 2, \tag{11}$$

it follows from (9), (10) and (11) that

$$d = 2 \quad \text{and} \quad b = -2. \tag{12}$$

Combining the results in (12) and (9) with (6) then yields

$$f(x, y) = 2 - 2y, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{13}$$

Hence, the equation of the equation for the plane containing the line $y = 1$ in the xy–plane and the line $z = 2$ in the xz–plane is

$$z = 2 - 2y.$$

A sketch of the graph of f given in (13) is shown in Figure 2. □

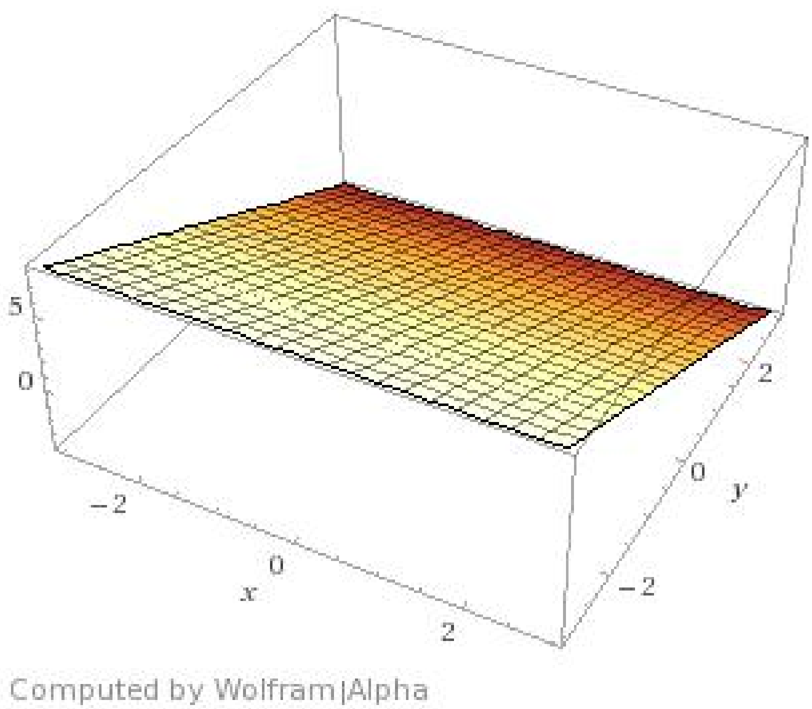

Figure 2: Sketch of Plane in Problem 2

3. An affine function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by the formula
$$f(x,y) = d + ax + by, \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{14}$$
where a, b and d are real numbers.

   Determine values for a, b and d so that the graph of $z = f(x,y)$ intersects the xz–plane in the line $z = 3x + 4$ and it intersects the yz–plane in the line $z = y + 4$.

   ***Solution**:* We are given that the section of the plane $z = f(x,y)$ in the xz–plane,
$$z = d + ax, \tag{15}$$
is the line
$$z = 3x + 4. \tag{16}$$
Thus, comparing (15) and (16),
$$a = 3 \quad \text{and} \quad d = 4. \tag{17}$$
Similarly, since the section of the plane in the yz–plane,
$$z = d + by, \tag{18}$$
is the line
$$z = y + 4, \tag{19}$$
by comparing (18) and (19), we get that
$$b = 1. \tag{20}$$
Putting together the results in (17) and (20), we obtain that
$$a = 3, \quad b = 1 \quad \text{and} \quad d = 4.$$
□

4. In each of the following, sketch the graph of $z = f(x,y)$ for the given affine function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$.

   (a) $f(x,y) = 2 - x - 2y$, for all $(x,y) \in \mathbb{R}^2$.
   ***Solution**:* A sketch of the plane $z = 2 - x - 2y$ is shown in Figure 3. □

   (b) $f(x,y) = 4 + x - 2y$, for all $(x,y) \in \mathbb{R}^2$.
   ***Solution**:* A sketch of the plane $z = 4 + x - 2y$ is shown in Figure 4. □

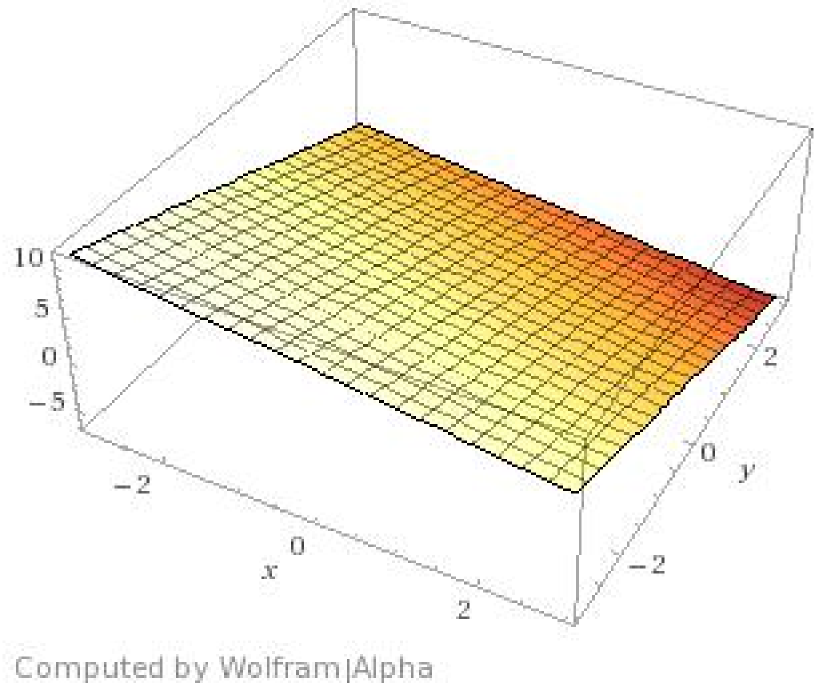

Figure 3: Sketch of Plane $z = 2 - x - 2y$

5. An affine function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by the formula

$$f(x, y) = d + ax + by, \qquad \text{for all } (x, y) \in \mathbb{R}^2, \tag{21}$$

where a, b and d are real numbers such that $b \neq 0$.

(a) Verify that the contour curves of f are lines of slope $-a/b$.

***Solution**:* The contour curves of the function f given in (21) are the graphs of the equations

$$d + ax + by = c,$$

or

$$ax + by = c - d,$$

for $c \in \mathbb{R}$; so that, if $b \neq 0$, the contour curves are the lines

$$y = -\frac{a}{b}x + \frac{c - d}{b},$$

which have slope $-a/b$. □

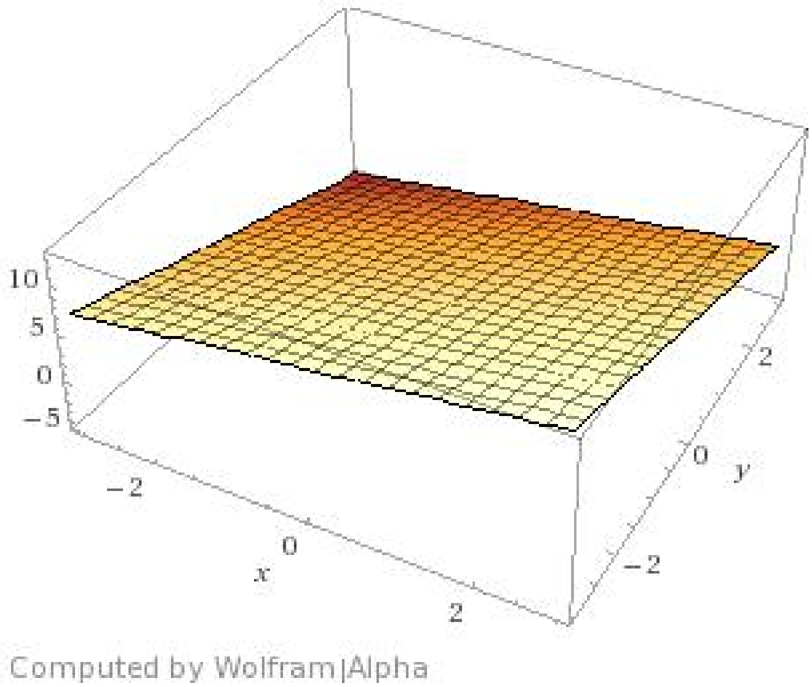

Figure 4: Sketch of Plane $z = 4 + x - 2y$

(b) Verify that $f(x + b, y - a) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.
***Solution**:* Use (21) to compute

$$\begin{aligned} f(x+b, y-a) &= d + a(x+b) + b(y-a) \\ &= d + ax + ab + by - ba \\ &= d + ax + by \\ &= f(x, y), \end{aligned}$$

which was to be shown. □

(c) Give an interpretation for the results in parts (a) and (b).
***Solution**:* The points (x, y) and $(x + b, y - a)$ lie on a line of slope $-a/b$; hence, the points (x, y) and $(x + b, y - a)$ lie on the same contour curve of the function f. Consequently, $f(x + b, y - a)$ and $f(x, y)$ have the same value. □