Solutions to Assignment #19

1. Compute the first partial derivatives of the function f given by

$$f(x,y)=\frac{x}{x^2+y^2}, \qquad \text{for } (x,y)\neq(0,0).$$

***Solution**:* Apply the Quotient Rule to compute, for $(x,y)\neq(0,0)$,

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}\\ &= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2};\end{aligned}$$

so that

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \text{for } (x,y)\neq(0,0).$$

Similarly, for $(x,y)\neq(0,0)$,

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2}\\ &= \frac{-2xy}{(x^2+y^2)^2};\end{aligned}$$

so that,

$$\frac{\partial f}{\partial y}(x,y) = -\frac{2xy}{(x^2+y^2)^2}, \qquad \text{for } (x,y)\neq(0,0).$$

□

2. Compute the first partial derivatives of the function f given by

$$f(x,y)=e^{-x}\sin y, \qquad \text{for all } (x,y)\in\mathbb{R}^2.$$

***Solution**:* Compute

$$\frac{\partial f}{\partial x}(x,y) = -e^{-x}\sin y, \qquad \text{for all } (x,y)\in\mathbb{R}^2, \tag{1}$$

and

$$\frac{\partial f}{\partial y}(x,y) = e^{-x}\cos y, \qquad \text{for all } (x,y)\in\mathbb{R}^2 \tag{2}$$

□

3. Find a function f of the variables x and y satisfying

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = y + 2x; \\ \\ \dfrac{\partial f}{\partial y}(x,y) = x, \end{cases} \tag{3}$$

for all $(x,y) \in \mathbb{R}^2$.

***Solution**:* Integrate on both sides of the first equation in (3) with respect to x to obtain that

$$f(x,y) = xy + x^2 + g(y), \tag{4}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function.

Next, differentiate with respect to y on both sides of (4) to get

$$\frac{\partial f}{\partial y} = x + g'(y). \tag{5}$$

Comparing (5) and the second equation in (3) we see that

$$g'(y) = 0, \quad \text{for all } y \in \mathbb{R},$$

from which we get that

$$g(y) = c \ \text{ (a constant)}, \qquad \text{for all } y \in \mathbb{R}. \tag{6}$$

It then follows from (4) and (6)

$$f(x,y) = xy + x^2 + c, \quad \text{for all } (x,y) \in \mathbb{R}^2,$$

where c is a constant. □

4. Let f be as in Problem 2.

Compute the second partial derivatives of f:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}.$$

***Solution**:* Differentiate on both sides of (1) with respect to x to get

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{-x} \sin y, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{7}$$

Differentiate on both sides of (1) with respect to y to get

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -e^{-x}\cos y, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

Differentiate on both sides of (2) with respect to x to get

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -e^{-x}\cos y, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

Differentiate on both sides of (2) with respect to y to get

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -e^{-x}\sin y, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{8}$$

□

5. Let $f(x,y) = e^{-x}\cos y$ for all $(x,y) \in \mathbb{R}^2$.

   Verify that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

***Solution**:* Compute the partial derivatives of f to get

$$\frac{\partial f}{\partial x}(x,y) = -e^{-x}\cos y, \qquad \text{for all } (x,y) \in \mathbb{R}^2, \tag{9}$$

and

$$\frac{\partial f}{\partial y}(x,y) = -e^{-x}\sin y, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{10}$$

Take the partial derivative with respect to x on both sides of (9) to get

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{-x}\cos y, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{11}$$

Take the partial derivative with respect to y on both sides of (10) to get

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -e^{-x}\cos y, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{12}$$

Add the results in (11) and (12) to get that

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^{-x}\cos y - e^{-x}\cos y = 0,$$

for all $(x,y) \in \mathbb{R}^2$, which was to be shown. □