Solutions to Assignment #1

Background and Definitions.
In Section 2.1 of the class lecture notes, we derived the Kermack–McKendrick SIR model for the spread of an infections disease in a population,

$$\begin{cases} \dfrac{dS}{dt} = -\beta S I; \\[2ex] \dfrac{dI}{dt} = \beta S I - \gamma I; \\[2ex] \dfrac{dR}{dt} = \gamma I. \end{cases} \tag{1}$$

The quantity $S(t)$ denotes the number of individuals in the population that are susceptible to getting the disease, $I(t)$ is the number of individuals that have contracted the disease and can infect individuals from the susceptible class, and $R(t)$ is the number of individuals in the population that have recovered and are immune to the disease. The positive parameters β and γ are called the infection rate and recovery rate, respectively.

1. Give units for the parameters β and γ in the SIR system in (1).

   Solution: Solving for β in the first equation in (1) we obtain

   $$\beta = -\frac{1}{SI}\frac{dS}{dt};$$

   thus, β has units of 1 per individual per unit of time.

   Similarly, solving for γ in the third equation in (1) yields

   $$\gamma = \frac{1}{I}\frac{dR}{dt};$$

   so that, γ is in units of 1 over unit of time. □

2. Let $N(t) = S(t) + I(t) + R(t)$ for all t. Use the equations in (1) to derive the differential equation

   $$\frac{dN}{dt} = 0.$$

   Deduce that $N(t)$ must be a constant function.

***Solution**:* Differentiate N with respect to time to obtain

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}. \tag{2}$$

Substituting the expressions for the derivatives on the right–hand of (1) into the right—hand side of (2) yields

$$\begin{aligned}\frac{dN}{dt} &= \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} \\ &= -\beta SI + \beta SI - \gamma I + \gamma I;\end{aligned}$$

so that

$$\frac{dN}{dt} = 0,$$

and, therefore, N must be constant. □

3. Let $S_o = S(t)$, the initial number of susceptible individuals in the population and put

$$R_o = \frac{\beta S_o}{\gamma}. \tag{3}$$

Give the units for R_o.

The constant R_o is called the reproduction number.

***Solution**:* It follows from the result of Problem 1 that β has units of 1 per individual per unit of time and γ is in units of 1 over unit of time. Consequently,

$$R_o = \frac{\beta S_o}{\gamma}$$

has no units. □

4. Assume that at time $t = 0$, there are is only one infectious individual in the population and no one in the population has acquired immunity. Let N denote the total number of individuals in the population.

   (a) Compute S_o in terms of N.

   ***Solution**:* Solving for $S(t)$ in $S(t) + I(t) + R(t) = N$ we obtain

   $$S(t) = N - I(t) - R(t);$$

   so that,

   $$S(0) = N - I(0) - R(0),$$

   which gives $S_o = N - 1$. □

(b) Give the reproduction number, R_o, in (3) in this situation.

Solution: $R_o = \dfrac{\beta(N-1)}{\gamma}$. □

5. Let R_o be as computed in Problem 4.

(a) Assume that $R_o > 1$, and determine the sign of $I'(0)$. What do you conclude in this case? Explain the reasoning leading to your conclusion.

Solution: Use the second equation in (1) to get that

$$I'(t) = \beta S(t)I(t) - \gamma I(t).$$

Thus,

$$I'(0) = \beta S(0)I(0) - \gamma I(0),$$

from which we get

$$\begin{aligned} I'(0) &= \beta S_o - \gamma \\ &= \gamma\left(\frac{\beta S_o}{\gamma} - 1\right); \end{aligned}$$

so that

$$I'(0) = \gamma(R_o - 1). \tag{4}$$

It follows from (4) that, if $R_o > 1$, then $I'(0) > 0$; so that, $I(t)$ will increase for $t > 0$ near 0. Hence, the disease will spread in this case. □

(b) Assume that $R_o < 1$, and determine the sign of $I'(0)$. What do you conclude in this case? Explain the reasoning leading to your conclusion.

Solution: Use the result in (4) to deduce that, if $R_o < 1$, then $I'(0) < 0$. Thus, in this case the disease will not spread. □