Assignment #20

Due on Wednesday, May 1, 2019

Read Section 6.2, on *Linear Approximations,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 6.3, on *Linear Approximations and Partial Derivatives,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 6.4, on *Partial Derivatives and the Gradient,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions.

- **Linear approximation of a real valued function of two variables**. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane, and let (x_o, y_o) denote a point in D. Suppose that the partial derivatives of f exist and are continuous in D. The linear approximation for f at (x_o, y_o) is the affine function $L\colon \mathbb{R}^2 \to \mathbb{R}$ given by

  $$L(x,y) = f(x_o,y_o) + \frac{\partial f}{\partial x}(x_o,y_o)\cdot(x-x_o) + \frac{\partial f}{\partial y}(x_o,y_o)\cdot(y-y_o), \quad \text{for } (x,y) \in \mathbb{R}^2.$$

  $L(x,y)$ approximates $f(x,y)$ when (x,y) is very close to (x_o,y_o). We write

  $$f(x,y) \approx f(x_o,y_o) + \frac{\partial f}{\partial x}(x_o,y_o)\cdot(x-x_o) + \frac{\partial f}{\partial y}(x_o,y_o)\cdot(y-y_o),$$

  for (x,y) in D sufficiently close to (x_o,y_o).

- **The gradient of a function of two variables**. Let $f\colon D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$. Suppose that the partial derivatives of f exist in D. The gradient of f, denoted by ∇f, is the vector valued function $\nabla f\colon D \to \mathbb{R}^2$ defined by $\nabla f(x,y) = \dfrac{\partial f}{\partial x}(x,y)\ \hat{i} + \dfrac{\partial f}{\partial y}(x,y)\ \hat{j}$, for $(x,y) \in D$.

Do the following problems

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \dfrac{1}{2}x^2 + 2y^2$, for $(x,y) \in \mathbb{R}^2$.

   (a) Compute the gradient of f for all $(x,y) \in \mathbb{R}^2$.

   (b) Give the linear approximation to f for (x,y) near $(2,1)$.

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \sqrt{x^2+y^2}, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

   (a) Give the linear approximation to f near the point $(3,4)$.

   (b) Use the linear approximation to f at $(3,4)$ to estimate $f(2.98, 4.01)$.

3. Assume that the temperature in an unevenly heated plate is given by $T(x,y)$ °C at every point (x,y) in the plate, where T is a function of two variables with continuous partial derivatives T_x and T_y. Assume that $T(2,1) = 135$ °C, and that the partial derivatives of T at $(2,1)$ have values $T_x(2,1) = 16$ and $T_y(2,1) = -15$. Estimate the temperature at the point $(2.04, 0.97)$.

4. **The Differential of** f. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane. Let $\sigma\colon I \to \mathbb{R}^2$ denote a differentiable path defined on some interval $I \subseteq \mathbb{R}$ show interval lies in D. Denote the differential of σ by

$$d\sigma = dx\hat{\imath} + dy\hat{\jmath}.$$

   The differential of f, denoted by df, is defined by the dot product of ∇f and $d\sigma$,

$$df = \nabla f \cdot d\sigma.$$

   (a) Give and expression for computing the differential of f in terms of the partial derivatives of f and the differentials dx and dy.

   (b) Given $f(x,y) = xy$, for all $(x,y) \in \mathbb{R}^2$ to compute df.

5. Let $p(A,D)$ denote the expression giving the real number π, where A denotes the area enclosed by a circle and D the diameter of the circle.

   (a) Give and expression of $p(A,D)$.

   (b) Compute the differential of p.

   (c) Assume that a percent error of 0.001 can be made when measuring the area enclosed by the circle, and a percent error of 0.0005 can be made when measuring the diameter. Use the differential computed in part (b) to estimate the error in computing π.