Solutions to Assignment #20

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \dfrac{1}{2}x^2 + 2y^2$, for $(x,y) \in \mathbb{R}^2$.

   (a) Compute the gradient of f for all $(x,y) \in \mathbb{R}^2$.
   ***Solution**:* The gradient of f at $(x,y) \in \mathbb{R}^2$ is

   $$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\ \hat{i} + \frac{\partial f}{\partial y}(x,y)\ \hat{j},$$

   where

   $$\frac{\partial f}{\partial x}(x,y) = x,$$

   and

   $$\frac{\partial f}{\partial y}(x,y) = 4y.$$

   Consequently,

   $$\nabla f(x,y) = x\ \hat{i} + 4y\ \hat{j}, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{1}$$

   □

   (b) Give the linear approximation to f for (x,y) near $(2,1)$.
   ***Solution**:* The linear approximation to f at $(2,1)$ is

   $$L(x,y) = f(2,1) + \nabla f(2,1)` \cdot ((x-2)\ \hat{i} + (y-1)\ \hat{j}, \quad \text{for } (x,y) \in \mathbb{R}^2,$$

   where, according to (1),

   $$\nabla f(2,1) = 2\ \hat{i} + 4\ \hat{j}.$$

   Consequently,

   $$L(x,y) = 4 + 2((x-2) + 4(y-1), \quad \text{for } (x,y) \in \mathbb{R}^2.$$

   □

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

   $$f(x,y) = \sqrt{x^2+y^2}, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

(a) Give the linear approximation to f near the point $(3,4)$.

Solution: The linear approximation to f at $(3,4)$ is

$$L(x,y) = f(3,4) + \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4), \quad \text{for } (x,y) \in \mathbb{R}^2,$$

where

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

for $(x,y) \neq (0,0)$. Thus,

$$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4), \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{2}$$

□

(b) Use the linear approximation to f at $(3,4)$ to estimate $f(2.98, 4.01)$.

Solution: Use (2) to estimate

$$\begin{aligned} f(2.98, 4.01) &\approx L(2.98, 4.01) \\ &= 5 + \frac{3}{5}(2.98-3) + \frac{4}{5}(4.01-4) \\ &= 5 - \frac{3}{5}(0.02) + \frac{4}{5}(0.01) \\ &= 5 - 0.012 + 0.008; \end{aligned}$$

so that,

$$f(2.98, 4.01) \approx 4.996$$

□

3. Assume that the temperature in an unevenly heated plate is given by $T(x,y)$ °C at every point (x,y) in the plate, where T is a function of two variables with continuous partial derivatives T_x and T_y. Assume that $T(2,1) = 135$ °C, and that the partial derivatives of T at $(2,1)$ have values $T_x(2,1) = 16$ and $T_y(2,1) = -15$. Estimate the temperature at the point $(2.04, 0.97)$.

Solution: The linear approximation to T at $(2,1)$ is

$$L(x,y) = T(2.1) + T_x(2,1)(x-1) + T_y(2,1)(y-1), \quad \text{for } (x,y) \in \mathbb{R}^2;$$

so that

$$L(x, y) = 135 + 16(x - 1) - 15(y - 1), \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

Consequently,

$$\begin{aligned} T(2.04, 0.97) &\approx L(2.04, 0.97) \\ &= 135 + 16(2.04 - 2) - 15(0.97 - 1) \\ &= 135 + 16(0.04) + 15(0.03) \\ &= 135 + 0.64 + 0.45; \end{aligned}$$

so that,

$$T(2.04, 0.97) \approx 136.09\ ^\circ\text{C}.$$

□

4. **The Differential of** f. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane. Let $\sigma\colon I \to \mathbb{R}^2$ denote a differentiable path defined on some interval $I \subseteq \mathbb{R}$ show interval lies in D. Denote the differential of σ by

$$d\sigma = dx\hat{i} + dy\hat{j}.$$

The differential of f, denoted by df, is defined by the dot product of ∇f and $d\sigma$,

$$df = \nabla f \cdot d\sigma.$$

(a) Give and expression for computing the differential of f in terms of the partial derivatives of f and the differentials dx and dy.

***Solution**:* Compute

$$\nabla f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j},$$

and

$$\begin{aligned} df &= \nabla f \cdot d\sigma \\ &= \left(\frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j}\right) \cdot (dx\hat{i} + dy\hat{j}); \end{aligned}$$

so that,

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy. \tag{3}$$

□

(b) Given $f(x, y) = xy$, for all $(x, y) \in \mathbb{R}^2$ to compute df.
***Solution**:* Use the formula in (3) with

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x,$$

to get

$$df = y\ dx + x\ dy.$$

□

5. Let $p(A, D)$ denote the expression giving the real number π, where A denotes the area enclosed by a circle and D the diameter of the circle.

(a) Give and expression of $p(A, D)$.
***Solution**:* Using the formula

$$A = \pi r^2,$$

for the area of a circle of radius r, we derive the expression

$$p(A, D) = \frac{4A}{D^2} \tag{4}$$

for computing π in terms of the area, A, enclosed by the circle and the diameter, D, of the circle. □

(b) Compute the differential of p.
***Solution**:* We compute the differential of p using the formula derived in (3) to get

$$dp = \frac{\partial p}{\partial A}\ dA + \frac{\partial p}{\partial D}\ dD,$$

where, using the definition of p in (4)

$$\frac{\partial p}{\partial A} = \frac{4}{D^2} \quad \text{and} \quad \frac{\partial p}{\partial D} = -\frac{8A}{D^3}, \quad \text{for } D > 0.$$

Thus,

$$dp = \frac{4}{D^2}\ dA - \frac{8A}{D^3}\ dD, \quad \text{for } D > 0. \tag{5}$$

□

(c) Assume that a percent error of 0.001 can be made when measuring the area enclosed by the circle, and a percent error of 0.0005 can be made when measuring the diameter. Use the differential computed in part (b) to estimate the error in computing π.

***Solution**:* The percent error in the estimation of π given by the expression in (4) can be estimated by

$$\frac{|dp|}{p}, \qquad \text{for } p > 0, \tag{6}$$

where, according to (5) and the triangle inequality,

$$|dp| \leqslant \frac{4}{D^2}\,|dA| + \frac{8A}{D^3}\,|dD|, \qquad \text{for } D > 0. \tag{7}$$

Next, use the definition of p in (4) and the estimate in (7) to obtain the following estimate for the percent error in (6):

$$\frac{|dp|}{p} \leqslant \frac{|dA|}{A} + 2\frac{|dD|}{D}, \qquad \text{for } A > 0 \text{ and } D > 0. \tag{8}$$

Applying (8) with

$$\frac{|dA|}{A} \leqslant 0.001 \quad \text{and} \quad \frac{|dD|}{A} \leqslant 0.0005,$$

we obtain that

$$\frac{|dp|}{p} \leqslant 0.002.$$

Thus, the percent error in the estimation of π using the formula in (4) is at most 0.002, or 0.2%. □