Assignment #21

Due on Friday, May 3, 2019

Read Section 6.4, on *Partial Derivatives and the Gradient,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 6.5, on *The Gradient and the Chain–Rule,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions.

- **The Chain–Rule**. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane, and let $\sigma\colon I \to \mathbb{R}^2$, for some open interval I, denote a differentiable path with $\sigma(t) \in D$ for all $t \in I$. Suppose that the partial derivatives of f exist and are continuous in D. Then, for any $t \in I$,

$$\frac{d}{dt}[f(\sigma(t))] = \nabla f(\sigma(t)) \cdot \sigma'(t),$$

  where ∇f denotes the gradient of f, $\sigma'(t)$ is the derivative of the path σ, and the dot between ∇f and σ' indicates the dot product of the two vectors.

- **Directional Derivative**. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D. Suppose that the first order partial derivatives of f exist at (x, y). Let $\hat{u}$ denote a unit vector in $\mathbb{R}^2$. The directional derivative of f at (x, y) in the direction of $\hat{u}$, denoted by $D_{\hat{u}}f(x, y)$, is defined by

$$D_{\hat{u}}f(x, y) = \nabla f(x, y) \cdot \hat{u},$$

  the dot product of the gradient of f at (x, y) with $\hat{u}$.

Do the following problems

1. Let $f(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$. Compute the directional derivative f $(2, 1)$ in the direction of the line $y = x$ towards the first quadrant.

   Suggestion: Find a unit vector $\hat{u}$ in the direction of the line $y = x$ towards the first quadrant.

2. The directional derivative of a function, f, of two variables, x and y, at $(2, 1)$ in the direction towards the point $(1, 3)$ is $-2/\sqrt{5}$, and the directional derivative at $(2, 1)$ in the direction of towards the point $(5, 5)$ is 1. Compute the first–order partial derivatives of f at $(2, 1)$.

3. A bug is moving on a two–dimensional plate, D, with temperature $u(x,y)$ for all $(x,y) \in D$. Assume that at $(x_o, y_o) \in D$,

$$\frac{\partial u}{\partial x}(x_o, y_o) = -2 \quad \text{and} \quad \frac{\partial u}{\partial y}(x_o, y_o) = 1.$$

Suppose the velocity of the bug at when it is at (x_o, y_o) is given by the vector $v = 4\hat{\imath} + 7\hat{\jmath}$. Compute the rate of change of temperature along the path of the bug at the point (x_o, y_o).

4. Let $\hat{u}$ denote a unit vector and put $\sigma(t) = x_o\hat{\imath} + y_o\hat{\jmath} + t\hat{u}$ for all $t \in \mathbb{R}$. Let $f \colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane that contains the point (x_o, y_o).

   (a) Apply the Chain Rule to compute $\dfrac{d}{dt}[f(\sigma(t))]$ at $t = 0$. Explain why this yields the directional derivative of f at (x_o, y_o) in the direction of $\hat{u}$.

   (b) Deduce that

$$D_{\hat{u}}f(x,y) = \|\nabla f(x,y)\| \cos\theta, \quad \text{for all } (x,y) \in D, \tag{1}$$

   where θ is the angle that $\nabla f(x,y)$ makes with the unit vector $\widehat{u}$.
   Conclude from (1) that the rate of change of f at (x,y) is the largest in the direction of the gradient of f at (x,y).

5. Let $f(x,y) = 3xy + y^2$ for all $(x,y) \in \mathbb{R}^2$.

   (a) Give the direction of maximum rate of change of f at $(2,3)$.

   (b) Give the direction in which f is decreasing the fastest at $(2,3)$.

   (c) Give the direction in which the rate of change of f is at $(2,3)$ is zero.