Solutions to Assignment #2

1. A curve C in the xy–plane is parametrized by the equations

$$x(t) = t + 2 \quad \text{and} \quad y(t) = -t + 1, \quad \text{for } t \in \mathbb{R}.$$

Sketch the graph of C.

***Solution**:* The parametric equations of C equations are

$$\begin{cases} x = 2 + t; \\ y = 1 - t. \end{cases} \tag{1}$$

Solving for t in the first equation in (1) and substituting into the second equation yields

$$y = 1 - (x - 2),$$

which can be rewritten as

$$y = 3 - x.$$

This is the equation of a straight line in $\mathbb{R}^2$ with intercepts $(3, 0)$ and $(0, 5)$. A sketch of the line is shown in Figure 1. □

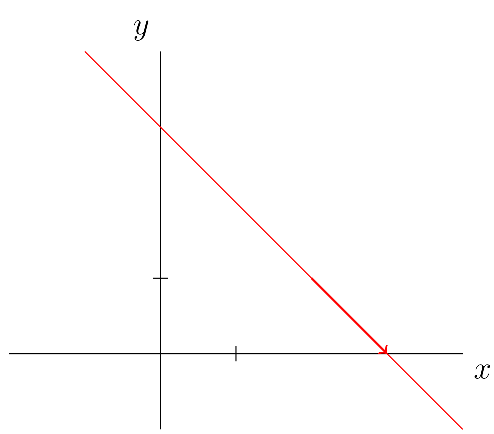

Figure 1: Sketch of Line in Problem 1

2. A curve C in the xy–plane is parametrized by the equations

$$x(t) = \cos t \quad \text{and} \quad y(t) = \sin t, \quad \text{for } 0 \leqslant t \leqslant \pi$$

Sketch the graph of C.

Solution: Eliminating the parameter t from the parametric equations

$$\begin{array}{rcl} x & = & \cos t; \\ y & = & \sin t, \end{array} \quad \text{for } 0 \leqslant t \leqslant \pi, \tag{2}$$

yields the equation

$$x^2 + y^2 = 1.$$

or Thus, C is the semicircle along the unit circle in the plane from the point $(1, 0)$ to the point $(-1, 0)$. A sketch of C is shown in Figure 2. □

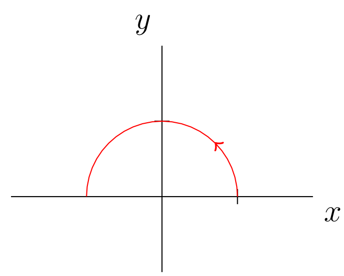

Figure 2: Sketch of Line in Problem 2

3. Suppose that $(x(t), y(t))$ solves the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & 2; \\ \\ \dfrac{dy}{dt} & = & 1, \end{cases} \tag{3}$$

subject to the initial conditions $x(0) = x_o$ and $y(0) = y_o$, for some given real numbers x_o and y_o.

Find $x(t)$ and $y(t)$, for all t, and sketch the graph of the parametrized curve that these functions determine.

Solution: Integrating the equations in (3) separately yields the parametric equations

$$\begin{array}{rcl} x & = & 2t + c_1; \\ y & = & t + c_2, \end{array} \tag{4}$$

where c_1 and c_2 are constants of integration. Substituting 0 for t in (4) and using the initial conditions yields $c_1 = x_o$ and $c_2 = y_o$, so that

$$\begin{aligned} x(t) &= x_o + 2t; \\ y(t) &= y_o + t. \end{aligned} \tag{5}$$

The graph of the curves parametrized by the equations in (5) is a straight line through the point $P(x_o, y_o)$ and slope 1/2. A sketch of a possible line is shown in Figure A sketch of the line is shown in Figure 3. The other lines are parallel to this line. □

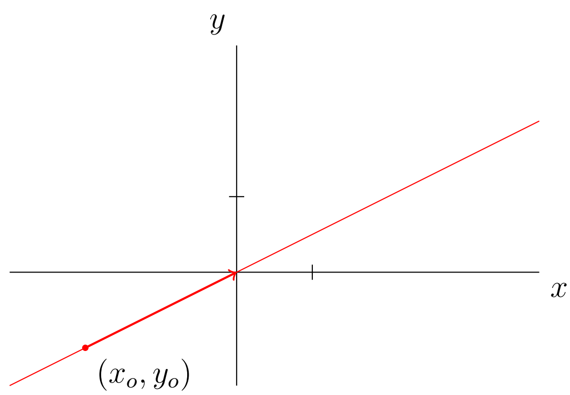

Figure 3: Sketch of Line in Problem 3

4. For each of the given parametrized curves, $(x(t), y(t))$, compute the derivatives $(x'(t), y'(t))$.

   (a) $(x(t), y(t)) = (t, t^2)$, for all $t \in \mathbb{R}$.
   Solution: $(x'(t), y'(t)) = (1, 2t)$, for all $t \in \mathbb{R}$. □

   (b) $(x(t), y(t)) = (t\cos t, t\sin t)$, for all $t \in \mathbb{R}$. ***Solution:*** Use the Product Rule to compute

$$(x'(t), y'(t)) = (\cos t - t\sin t, \sin t + t\cos t), \qquad \text{for all } t \in \mathbb{R}.$$

□

5. Given that $(x'(t), y'(t)) = (1, 2t)$, for all t, and that $(x(0), y(0)) = (1, 1)$, compute $(x(t), y(t))$, for all $t \in \mathbb{R}$, and sketch the graph of the parametrized curve.

   ***Solution**:* Integrate the equations

$$\begin{cases} \dfrac{dx}{dt} = 1; \\ \\ \dfrac{dy}{dt} = 2t \end{cases} \tag{6}$$

separately to get

$$\begin{array}{rcl} x(t) & = & t + c_1; \\ y(t) & = & t^2 + c_2, \end{array} \quad \text{for } t \in \mathbb{R}, \tag{7}$$

where c_1 and c_2 are constants of integration. Substituting 0 for t in (7) and using the initial conditions yields $c_1 = 1$ and $c_2 = 1$, so that

$$\begin{array}{rcl} x(t) & = & t + 1; \\ y(t) & = & t^2 + 1, \end{array} \quad \text{for } t \in \mathbb{R}. \tag{8}$$

Eliminating the parameter t from the parametric equations in (8) yields the equation

$$y = (x - 1)^2 + 1, \tag{9}$$

which is the equation of a parabola in the xy–plane with vertex at $(1, 1)$. A sketch of the graph in equation (9) is shown in Figure 4. □

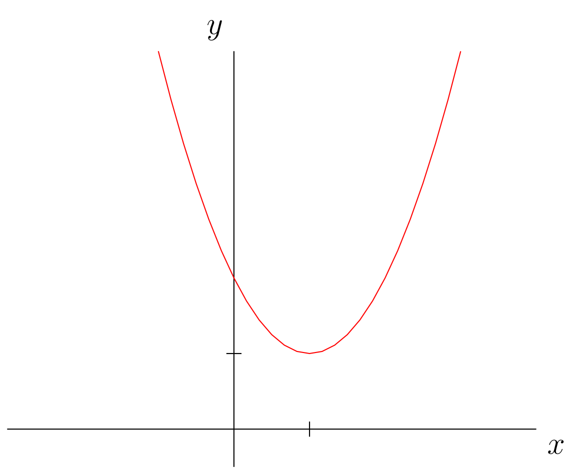

Figure 4: Sketch of parabola in Problem 5