Solutions to Assignment #4

1. Let the points P and Q in $\mathbb{R}^2$ have coordinates $(1, -1)$ and $(-2, 3)$, respectively.

   (a) Sketch the displacement vector $\overrightarrow{PQ}$.
   Solution: See sketch in Figure 1. $\square$

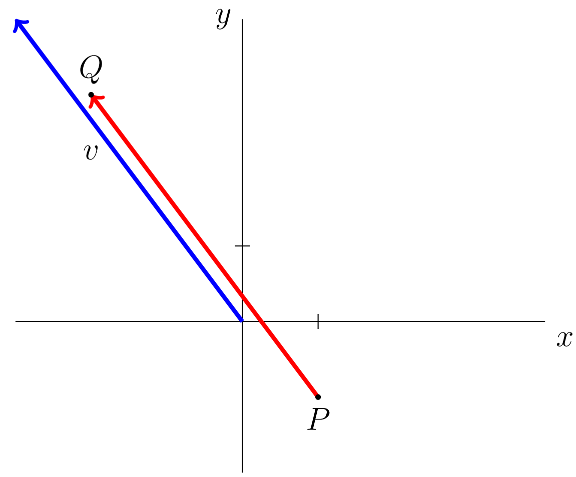

Figure 1: Sketch of directed line segment from P to Q

   (b) Sketch the vector $v = \overrightarrow{PQ}$ in standard position.
   Solution: See sketch of v in standard position in Figure 1. $\square$

   (c) Compute the cosine of the angle that v makes with the positive x–axis.
   Solution: Write

$$v = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

   Let θ denote the angle that v (in standard position) makes with the positive x–axis. Then,

$$\cos\theta = \frac{-3}{\|v\|},$$

   where

$$\|v\| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

   Thus,

$$\cos\theta = -\frac{3}{5}.$$

$\square$

(d) Compute the norm, $\|v\|$, of the vector v in part (a) and find a vector, $\widehat{u}$, of norm 1 that is in the same direction as the vector v.

Solution: The norm of v was computed to be $\|v\| = 5$ in the previous part.

Set

$$\widehat{u} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}.$$

Then, $\|\widehat{u}\| = 1$ and $\widehat{u}$ is in the same direction as that of v. □

2. Let P, Q and v be as in Problem 1.

(a) Give the parametric equations of the line through the points P and Q.

Solution: The parametric equations of the line through P and Q are

$$\begin{cases} x = -3t + 1; \\ y = 4t - 1, \end{cases} \quad \text{for } t \in \mathbb{R}.$$

□

(b) Give the parametric equations of the line through P that is perpendicular to the line found in part (a).

Solution: The slope of the line found in part (a) is

$$m = \frac{4}{-3} = -\frac{4}{3}.$$

Thus, the slope of a line that is perpendicular to the line through P and Q is

$$-\frac{1}{m} = \frac{3}{4}.$$

Thus, the equation of the line through P that is perpendicular to the line through P and Q is

$$y = \frac{3}{4}(x - 1) - 1. \quad (1)$$

Hence, making the parametrization

$$x = 4t + 1, \quad \text{for } t \in \mathbb{R}, \quad (2)$$

we get from (1) that

$$y = 3t - 1. \quad (3)$$

Combining (2) and (3) yields the parametrization

$$\begin{cases} x = 4t + 1; \\ y = 3t - 1, \end{cases} \quad \text{for } t \in \mathbb{R}.$$

□

(c) Give a vector, w, that is perpendicular to v and such that $\|w\| = 1$.

Solution: Let

$$w = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}.$$

Then, $\|w\| = 1$ and w is perpendicular to v because it is parallel to a line perpendicular to v. □

3. Let v denote the vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$. For a real number c, the scalar multiple cv of v is defined by $cv = \begin{pmatrix} ca \\ cb \end{pmatrix}$.

(a) Suppose that $c \neq 0$. Explain why the vector cv lies in the same line through the origin as the vector v. Discuss the cases $c > 0$ and $c < 0$.

Solution: We consider the set of scalar multiples of v:

$$L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix},\ t \in \mathbb{R} \right\}. \tag{4}$$

We assume that $a > 0$ and $b > 0$.

A vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is in L, according to the definition of L in (4), if and only if

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at \\ bt \end{pmatrix}, \quad \text{for some } t \in \mathbb{R},$$

from which we get the parametric equations

$$\begin{cases} x = at; \\ y = bt, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{5}$$

The equations in (5) are a parametrization of a straight line through the origin $(0, 0)$ and the point (a, b) in $\mathbb{R}^2$. Thus, L is a straight line in the direction of the vector v. This is shown in Figure 2. Hence, all the multiples of v lie in a line through the origin along the vector v; that is, the line

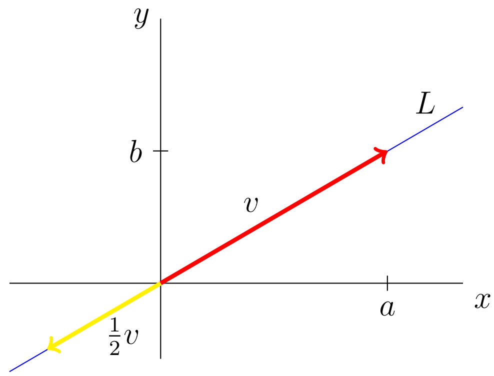

Figure 2: Line generated by v

through the points $(0,0)$ and (a,b). We note that, if $t>0$, tv lies along the direction of v; and, if $t<0$, tv points in the opposite direction to that of v. The sketch in Figure 2 shows the vector $-\dfrac{1}{2}v$, for the case in which both a and b are assumed to be positive. □

(b) Use the definition of the norm of vectors to verify that $\|cv\| = |c|\ \|v\|$, where $|c|$ is the absolute value of c.

Solution: Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$. Then, $cv = \begin{pmatrix} ca \\ cb \end{pmatrix}$; so that,

$$\begin{aligned} \|cv\| &= \sqrt{(ca)^2 + (cb)^2} \\ &= \sqrt{c^2a^2 + c^2b^2} \\ &= \sqrt{c^2(a^2+b^2)} \\ &= \sqrt{c^2}\sqrt{a^2+b^2}. \end{aligned}$$

Thus, using the definition of the norm of v and the fact that $\sqrt{c^2} = |c|$, the absolute value of c, we get that

$$\|cv\| = |c|\|v\|, \tag{6}$$

which was to be shown. □

(c) Suppose that $\|v\| \neq 0$ and put $c = \dfrac{1}{\|v\|}$. Compute $\|cv\|$. What do you conclude?

***Solution**:* Using the result in (6), compute

$$\begin{aligned}\|cv\| &= \left\| \frac{1}{\|v\|}\; v \right\| \\ &= \left| \frac{1}{\|v\|} \right| \|v\| \\ &= \frac{1}{\|v\|}\|v\| \\ &= 1.\end{aligned}$$

Thus, cv is a unit vector. □

4. Let J denote and open interval of real numbers and $\sigma\colon J \to \mathbb{R}^2$ denote a differeantiable path given by

$$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \text{for } t \in J.$$

Assume that $\|\sigma(t)\| \neq 0$ for all $t \in \mathbb{R}$, and define the real–value function $f\colon J \to \mathbb{R}$ by

$$f(t) = \|\sigma(t)\|, \qquad \text{for } t \in J.$$

Use the Chain Rule to show that f is differentiable and compute $f'(t)$ for all $t \in J$. Give a formula for computing $f'(t)$, for all $t \in J$, in terms of $x(t)$, $y(t)$, $x'(t)$, $y'(t)$, and $\|\sigma(t)\|$.

***Solution**:* Compute

$$f(t) = \sqrt{(x(t))^2 + (y(t))^2}, \qquad \text{for } t \in J.$$

Then, since $(x(t))^2 + (y(t))^2 > 0$ for all $t \in J$, f is the composition of two differentiable functions. Hence, by the Chain Rule, f is differentiable and

$$f'(t) = \frac{1}{2\sqrt{(x(t))^2 + (y(t))^2}} \cdot \frac{d}{dt}\left[(x(t))^2 + (y(t))^2\right];$$

so that, applying the Chain Rule again,

$$\begin{aligned} f'(t) &= \frac{1}{2\sqrt{(x(t))^2 + (y(t))^2}} \cdot [2x(t)x'(t) + 2y(t)y'(t)] \\ &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{(x(t))^2 + (y(t))^2}}; \end{aligned}$$

or, using the definition of the norm of $\sigma(t)$,

$$f'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{\|\sigma(t)\|}, \qquad \text{for } t \in J. \tag{7}$$

We can rewrite (7) in terms of the dot product of $\sigma(t)$ and $\sigma'(t)$:

$$f'(t) = \frac{\sigma(t) \cdot \sigma'(t)}{\|\sigma(t)\|}, \qquad \text{for } t \in J. \tag{8}$$

□

5. Let P and Q denote points in the xy–plane with Cartesian coordinates $(1, 0)$ and $(0, 1)$, respectively.

   (a) Give the equation of the line through P and Q in Cartesian coordinates.
   ***Solution**:* The equation of the line through P and Q, in Cartesian coordinates, is

   $$x + y = 1,$$

   or

   $$y = 1 - x. \tag{9}$$

   □

   (b) Give parametric equations of the line through P and Q.
   ***Solution**:* Use the equation in (9) and the parametrization $x = t$, for $t \in \mathbb{R}$, to get

   $$\begin{cases} x &= t; \\ y &= 1 - t, \end{cases} \qquad \text{for } t \in \mathbb{R}. \tag{10}$$

   □

   (c) Let

   $$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

   be the parametrization of the line through P and Q that you found in part (b).
   Define $f(t) = \|\sigma(t)\|$, for all $t \in \mathbb{R}$.
   Find the value of t in $\mathbb{R}$ for which $f(t)$ is the smallest possible. Use this fact to find the point on the line through P and Q that is the closest to the origin in $\mathbb{R}^2$. Explain the reasoning leading to your answer.

***Solution**:* Using the parametric equations in (10) we get that

$$\sigma(t) = \begin{pmatrix} t \\ 1-t \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{11}$$

To find the value of $t \in \mathbb{R}$ for which $f(t) = \|\sigma(t)\|$, for all $t \in \mathbb{R}$, is the smallest possible, we first find t for which $f'(t) = 0$, where $f'(t)$ is given by (7), or (8).

Now, $f'(t) = 0$ when the numerator in (7), or (8), is 0. Using (7), we get that $f'(t) = 0$ when

$$x(t)x'(t) + y(t)y'(t) = 0,$$

where

$$x(t) = t \quad \text{and} \quad y(t) = 1 - t;$$

so that,

$$x'(t) = 1 \quad \text{and} \quad y'(t) = -1.$$

We then have that $f'(t) = 0$ when

$$t(1) + (1-t)(-1) = 0,$$

or

$$t - 1 + t = 0,$$

or

$$2t = 1,$$

from which we get that $t = \dfrac{1}{2}$.

Thus, the point on the line through P and Q that is closest to the origin corresponds to

$$\sigma(1/2) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

□