Assignment #6

Due on Monday, February 18, 2019

Read Section 4.1, on *Vectors in the Plane,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions.
Dot product. Given vectors $v = x_1\widehat{i} + y_1\widehat{j}$ and $w = x_2\widehat{i} + y_2\widehat{j}$, the **dot product** of v and w, denoted by $v \cdot w$, is given by $v \cdot w = x_1x_2 + y_1y_2$.

Dot product and angle between vectors. If θ denotes the angle between the vectors v and w, then the dot product of v and w is also given by $v\cdot w = \|v\|\|w\|\cos\theta$, where $\|v\|$ and $\|w\|$ are the Euclidean norms of v and w, respectively.

Orthogonality. Vectors v and w in $\mathbb{R}^2$ are said to be **orthogonal**, or perpendicular, to each other if $v \cdot w = 0$.

Do the following problems

1. Let $v = a\widehat{i} + b\widehat{j}$ be a vector in $\mathbb{R}^2$ such that $\|v\| \neq 0$.

   (a) Give a vector $w \in \mathbb{R}^2$ that is orthogonal to v.

   (b) Give unit vectors $\widehat{v}$ and $\widehat{w}$ that are orthogonal to each other and such that $\widehat{v}$ is parallel to v and $\widehat{w}$ is parallel to w.

   (c) Let $\widehat{v}$ and $\widehat{w}$ be as in part (b). Put $u = c_1\widehat{v} + c_2\widehat{w}$, for some real numbers c_1 and c_2. Verify that
   $$\|u\|^2 = c_1^2 + c_2^2.$$
   Give and interpretation of this result.

2. Let v and w denote vectors in $\mathbb{R}^2$.

   (a) Use the fact that $|\cos\theta| \leqslant 1$ for all $\theta \in \mathbb{R}$ to show that
   $$|v \cdot w| \leqslant \|v\|\|w\|. \tag{1}$$
   The statement in (1) is called the Cauchy–Schwarz inequality.

   (b) Determine conditions on the vectors v and w under which equality occurs in (1). Explain the reasoning leading to your answer.

3. Use the Cauchy–Schwarz inequality in (1) to derive the **triangle inequality**:

$$\|v + w\| \leqslant \|v\| + \|w\|. \tag{2}$$

Suggestion: Compute $\|v + w\|^2 = (v + w) \cdot (v + w)$ using the properties of the dot product. Then, apply the Cauchy–Schwarz inequality.

4. Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

   (a) Explain why v and w are orthogonal.

   (b) Give unit vectors $\widehat{v}$ and $\widehat{w}$ that are orthogonal to each other and such that $\widehat{v}$ is parallel to v and $\widehat{w}$ is parallel to w.

   (c) Given any vector $u = a\widehat{i} + b\widehat{j}$, find c_1 and c_2, in terms of a and b, such that

   $$u = c_1\widehat{v} + c_2\widehat{w}.$$

   c_1 is called the component of u along the direction of v and c_2 is the component of u along the direction of w.

5. Let J denote an open interval of real numbers, and let $\sigma\colon J \to \mathbb{R}^2$ and $\gamma\colon J \to \mathbb{R}^2$ be differentiable paths given by

$$\sigma(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \text{and} \quad \gamma(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for } t \in J.$$

   (a) Define $f(t) = \sigma(t) \cdot \gamma(t)$, for $t \in J$. Use the definition of the dot product and the product rule to show that f is differentiable and give a formula for computing $f'(t)$.

   (b) Suppose that $\|\sigma(t)\| = C$, for all $t \in J$, and some constant C. Show that $\sigma'(t)$ is orthogonal to $\sigma(t)$ for all $t \in J$.

   Suggestion: Write $\|\sigma(t)\|^2 = C^2$ in terms of the dot product to get

   $$\sigma(t) \cdot \sigma(t) = C^2, \qquad \text{for all } t \in J. \tag{3}$$

   Take the derivative with respect to t on both sides of the equation in (3) and use the result derived in part (a).