Solutions to Assignment #8

1. Sketch the flow of the vector field

$$F(x,y) = -x\hat{\imath} + y\hat{\jmath}.$$

***Solution**:* The flow of the given vector field are curves that are obtained as solutions to the differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & -x; \\ & & \\ \dfrac{dy}{dt} & = & y, \end{cases} \tag{1}$$

Solutions of the equations in (1) are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{t} \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{2}$$

where c_1 and c_2 are arbitrary constants.

We will now proceed to sketch all types of solution curves determined by (2). These are determined by values of the parameters c_1 and c_2. For instance, when $c_1 = c_2 = 0$, (2) yields the equilibrium solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}.$$

We therefore obtain the equilibrium point $(0, o)$ sketched in Figure 1.

Next, if $c_1 \neq 0$ and $c_2 = 0$, then the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ 0 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

will lie in the positive x–axis, if $c_1 > 0$, or in the negative x–axis if $c_1 < 0$. These two possible trajectories are shown in Figure 1. The figure also shows the trajectories tending towards the origin, as indicated by the arrows pointing towards origin. The reason for this is that, as t increases, the exponential e^{-t} decreases.

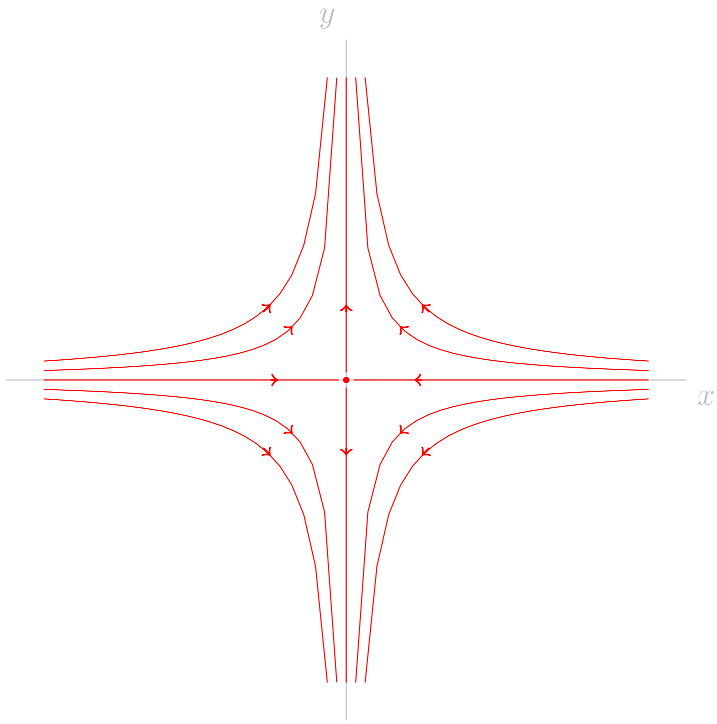

Figure 1: Sketch of Flow Field in Problem 1

Similarly, for the case $c_1 = 0$ and $c_2 \neq 0$, the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 e^t \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

will lie in the positive y–axis, if $c_2 > 0$, or in the negative y–axis if $c_2 < 0$. In this case, the trajectories point away from the origin because the exponential e^t increases as t increases.

The other flow field curves correspond to the case in which $c_1 \cdot c_2 \neq 0$. To see what these curves look like, combine the two parametric equations of the curves,

$$\begin{cases} x = c_1 e^{-t}; \\ y = c_2 e^t, \end{cases} \tag{3}$$

into a single equation involving x and y by eliminating the parameter t. This can be done by multiplying the equations in (3) to get

$$xy = c_1 c_2,$$

or

$$xy = c, \tag{4}$$

where we have written c for the product c_1c_2. The graphs of the equations in (4) are hyperbolas for $c \neq 0$. A few of these hyperbolas are sketched in Figure 1. Observe that all the hyperbolas in the figure have directions associate with them indicated by the arrows. The directions can be obtained from the formula for the solution curves in (2) or from the differential equations in the system in (1). For instance, in the first quadrant ($x > 0$ and $y > 0$), we get from the differential equations in (1) that $x'(t) < 0$ and $y'(t) > 0$ for all t; so that, the values of x along the trajectories decrease, while the y–values increase. Thus, the arrows point up and to the left as shown in Figure 1. □

2. Verify that the parametric equations

$$\begin{array}{rcl} x(t) & = & A\cos(t+\phi); \\ y(t) & = & A\sin(t+\phi), \end{array} \qquad \text{for } t \in \mathbb{R}, \tag{5}$$

where A and ϕ are constants, are the flow of the vector field

$$F(x,y) = -y\widehat{i} + x\widehat{j}. \tag{6}$$

Sketch the flow of the field.

***Solution**:* Taking the derivative with respect to t of the first function in (5), we obtain

$$\frac{dx}{dt} = -A\sin(t+\phi);$$

so that, in view of the second equation in (5),

$$\frac{dx}{dt} = -y. \tag{7}$$

Similarly, taking the derivative with respect to t of the second function in (5) yields

$$\frac{dy}{dt} = A\cos(t+\phi),$$

or

$$\frac{dy}{dt} = x, \tag{8}$$

by virtue of the first equation in (5).

The results in (7) and (8) show that the curves defined by the parametric equations in (5) constitute the flow of the vector field in (6). A sketch of this flow

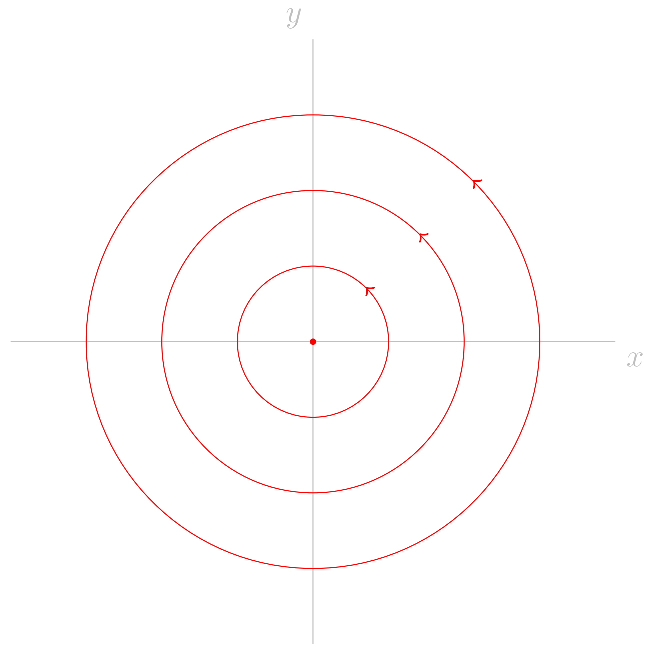

Figure 2: Sketch of Flow Field in Problem 2

field is shown in Figure 2. Note that the equations in (5) are parametrizations of circles of radius A centered at the origin; the origin corresponds to the case $A = 0$ in (5). □

3. Compute the flow of the field

$$F(x, y) = a\widehat{i} + b\widehat{j}, \tag{9}$$

where a and b are constants, and sketch it.

***Solution**:* The flow of the vector field in (9) are curves that are obtained as solutions to the differential equations

$$\left\{\begin{array}{rcl}\dfrac{dx}{dt} & = & a;\\[2ex] \dfrac{dy}{dt} & = & b,\end{array}\right.$$

which can be solved to yield

$$\begin{array}{rcl} x(t) & = & at + c_1; \\ y(t) & = & bt + c_2, \end{array} \qquad \text{for } t \in \mathbb{R}, \tag{10}$$

where c_1 and c_2 are constants of integration.

The equations in (10) are parametric equations of straight lines in the direction of the vector $a\hat{i} + b\hat{j}$. A few of these lines are sketched in Figure 3 for the case in which both a and b are positive. $\square$

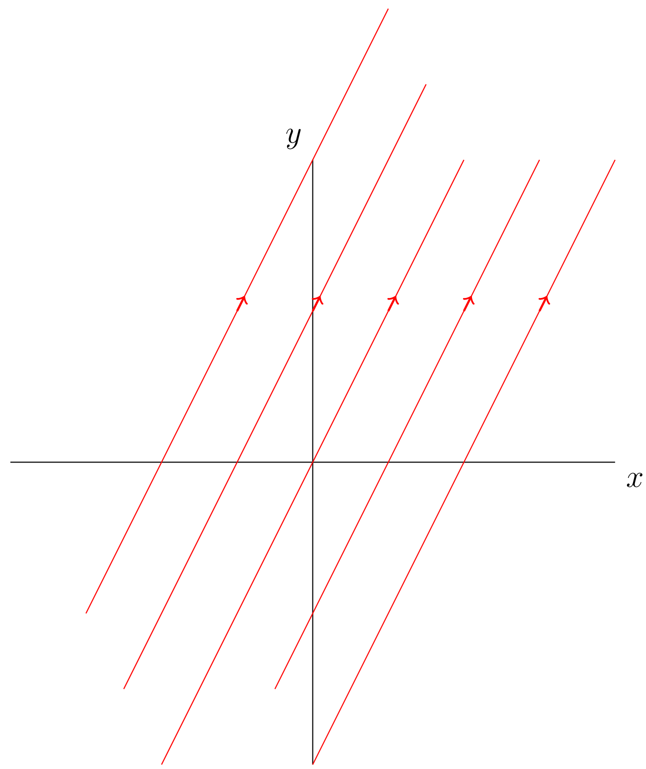

Figure 3: Sketch of Flow Field in Problem 3 for $a > 0$ and $b > 0$

4. Compute the flow of the field

$$F(x, y) = x\hat{i}, \tag{11}$$

and sketch it.

***Solution**:* The flow of the vector field in (11) are curves that are obtained as solutions to the differential equations

$$\begin{cases} \dfrac{dx}{dt} = x; \\[2ex] \dfrac{dy}{dt} = 0, \end{cases}$$

which can be solved to yield

$$\begin{array}{rcl} x(t) & = & c_1 e^t; \\ y(t) & = & c_2, \end{array} \qquad \text{for } t \in \mathbb{R}, \tag{12}$$

where c_1 and c_2 are constants of integration.

The equations in (10) are parametric equations of half lines parallel to the x–axis. The half lines are at level $y = c_2$ and point away from the y–axis because the exponential e^t increases with increasing t.

A few of these lines are sketched in Figure 4 for the case in which both a and b are positive. We observe that all points on the y–axis are also part of the flow

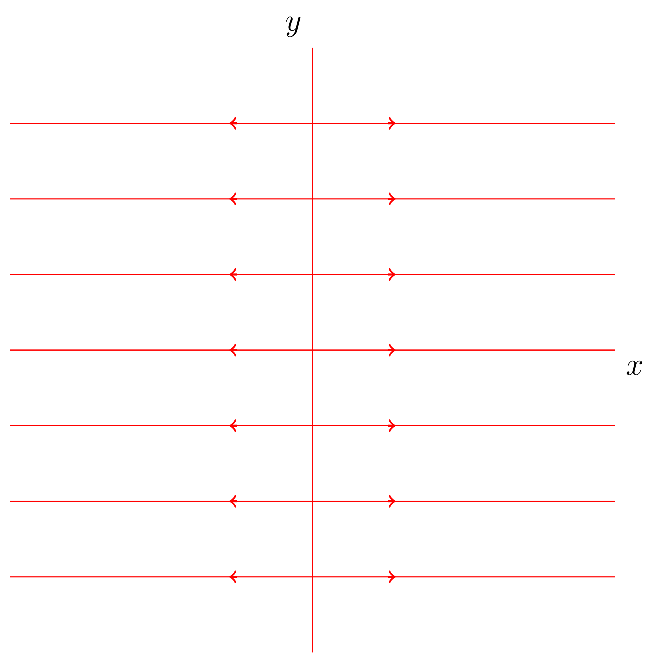

Figure 4: Sketch of Flow Field in Problem 4

field. These correspond to the case $c_1 = 0$ in (12). □

5. Verify that the parametric equations

$$\begin{array}{rcl} x(t) & = & a(e^t + e^{-t}); \\ y(t) & = & a(e^t - e^{-t}), \end{array} \qquad \text{for } t \in \mathbb{R}, \tag{13}$$

where a is a constant, are the flow of the vector field

$$F(x,y) = y\hat{\imath} + x\hat{\jmath}. \tag{14}$$

Sketch the flow of the field.

***Solution**:* Taking the derivative with respect to t of the first function in (13), we obtain

$$\frac{dx}{dt} = a(e^t - e^{-t});$$

so that, in view of the second equation in (13),

$$\frac{dx}{dt} = y. \tag{15}$$

Similarly, taking the derivative with respect to t of the second function in (13) yields

$$\frac{dy}{dt} = a(e^t + e^{-t}),$$

or

$$\frac{dy}{dt} = x, \tag{16}$$

by virtue of the first equation in (13).

The results in (15) and (16) show that the curves defined by the parametric equations in (13) constitute the flow of the vector field in (14).

In order to sketch the curves parametrized by the functions in (13), assume first that $a \neq 0$ and divide both equations by a, and then square them to get

$$\begin{aligned} \frac{x^2}{a^2} &= e^{2t} + 2 + e^{-2t}; \\ \frac{y^2}{a^2} &= e^{2t} - 2 + e^{-2t}. \end{aligned} \tag{17}$$

Next, subtract the second equation in (17) from the first to get

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 4,$$

which we can rewrite as

$$\frac{x^2}{4a^2} - \frac{y^2}{4a^2} = 1. \tag{18}$$

The graphs of equations in (18) are hyperbolas with x–intercepts at $\pm 2a$ and asymptotes $y = \pm x$. A few of these curves are sketched in Figure 5. The directions along the curves in Figure 5 were determined by looking at the signs of the derivatives of x and y determined by the expressions in (15) and (16). $\square$

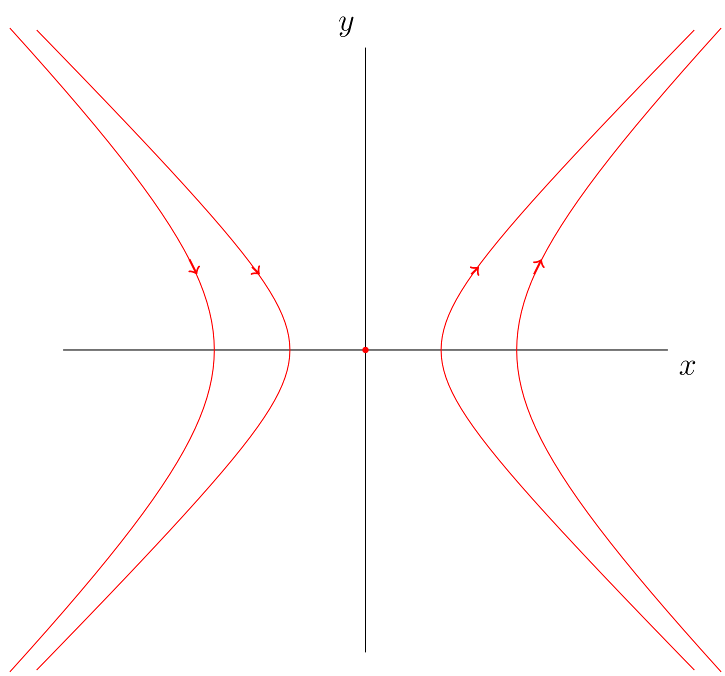

Figure 5: Sketch of Flow in Problem 5