Assignment #9

Due on Friday, March 8, 2019

Read Section 4.2.2, on *The Flow of a Vector Field,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions.
Time-Derivative Notation Convention. Given differentiable functions $x\colon J \to \mathbb{R}$ and $y\colon J \to \mathbb{R}$, we denote the derivative of x and y with respect to t by $\dot{x}$ and $\dot{y}$, respectively; so that,

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \dot{y} = \frac{dy}{dt}.$$

We shall reserve the prime notation to denote the derivative with respect to x; so that,

$$y' = \frac{dy}{dx} \quad \text{and} \quad x' = 1.$$

Do the following problems

1. In this problem, you will sketch the flow of the vector field

$$F(x, y) = y\hat{i} + x\hat{j}, \quad \text{for all } (x, y) \in \mathbb{R}^2. \tag{1}$$

The flow of the vector field in (1) are the solution curves of the system of differential equations

$$\begin{cases} \dot{x} = y; \\ \dot{y} = x. \end{cases} \tag{2}$$

(a) Use the expression

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad \text{for } \dot{x} \neq 0,$$

and the differential equations in (2) to obtain a differential equation involving only the variables x and y.

(b) Use separation of variables to solve the differential equations derived in part (a).

(c) Sketch all possible solution curves obtained in part (b).

(d) Indicate the directions along the solution curves of the system in (2) in the sketch obtained in part (c).

2. **The Hyperbolic Functions**. The hyperbolic cosine function, denoted by cosh, is the function $\cosh\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \text{for } t \in \mathbb{R}; \tag{3}$$

and the hyperbolic sine function, denoted $\sinh\colon \mathbb{R} \to \mathbb{R}$, is defined by

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \text{for } t \in \mathbb{R}. \tag{4}$$

Let $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$, for all $t \in \mathbb{R}$, where cosh and sinh are defined in (3) and (4), respectively.

(a) Verify that $\dot{x} = y$ and $\dot{y} = x$.

(b) Verify that $x^2 - y^2 = 1$.

(c) Sketch the curve parametrized by

$$\sigma(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \qquad \text{for all } t \in \mathbb{R}.$$

Indicate the direction given by the parametrization in the sketch.

(d) Give the equation of the tangent line to the curve at the point $(1, 0)$.

3. Consider a differentiable path $\sigma\colon J \to \mathbb{R}^2$ given by $\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, for $t \in J$, where J is an open interval

Let $r(t)$ denote the norm of $\sigma(t)$ for all $t \in J$ and $\theta(t)$ denote the angle that $\sigma(t)$ makes with the positive x–axis.

(a) Give formulas for computing $r(t)$ and $\theta(t)$, for $t \in J$, in terms of $x(t)$ and $y(t)$ for $t \in J$.

(b) Explain why the equations

$$\left\{ \begin{array}{rcl} x(t) & = & r(t)\cos(\theta(t)); \\ y(t) & = & r(t)\sin(\theta(t)), \end{array} \right. \qquad \text{for } t \in J, \tag{5}$$

are true.

4. Let σ, r and θ be as defined in Problem 3.

   Assume that $\sigma(t)$ is not the zero vector for all $t \in J$. Use the formulas in derived in Problem 3 to explain why r and θ are differentiable functions of t, and verify that

$$\begin{cases} \dot{r} &= \dfrac{\dot{x}}{r} \cdot x + \dfrac{\dot{y}}{r} \cdot y, \\ \\ \dot{\theta} &= \dfrac{\dot{y}}{r^2} \cdot x - \dfrac{\dot{x}}{r^2} \cdot y. \end{cases} \tag{6}$$

   Suggestion: Begin with the equations $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$; differentiate on both sides with respect to t; and apply the Chain Rule.

5. In this problem we find the solutions of the system

$$\begin{cases} \dot{x} &= -\beta y; \\ \\ \dot{y} &= \beta x, \end{cases} \tag{7}$$

   where $\beta > 0$.

   (a) Assume the equations in (7) are true, and use the equations in (6) to obtain a system of the form

$$\begin{cases} \dot{r} &= f(r, \theta); \\ \\ \dot{\theta} &= g(r, \theta), \end{cases} \tag{8}$$

   for some functions f and g that depend on r and θ.

   (b) Solve the system in (8).

   (c) Use the solutions obtained in part (b) and the equations in (5) to obtain solutions of the system in (7).