Review Problems for Exam 2

1. Compute and sketch the flow of the vector field

$$F(x,y) = -2x\hat{\imath} + y\hat{\jmath}, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

2. Compute and sketch the flow of the vector field

$$F(x,y) = -2x\hat{\imath} - 2y\hat{\jmath}, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

3. A particle of unit mass is moving along a path in the xy–plane parametrized by $\sigma(t) = R\sin(\omega t)\hat{\imath} + R\cos(\omega t)\hat{\jmath}$, for $t \in \mathbb{R}$, where R is measured in meters, t is measured in seconds, and ω in radians per second.

   The particle flies of its path on a tangent line at time t_o such that $\omega t_o = \dfrac{\pi}{3}$ radians.

   (a) Give the position and velocity of the particle at time t_o.

   (b) Give the equation of the path of the particle after it flies off its circular path.

   (c) Find the time $t > t_o$, if any, at which the particle meets the x–axis. Give the location of the particle at that time.

4. A particle moving in a straight line (along the x–axis) is moving according to the law of motion

$$\ddot{x} = 8x - 2\dot{x}. \tag{1}$$

   Define

$$x(t) = e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{2}$$

   (a) Determine distinct values of λ for which the function x defined in (2) solves the differential equation in (1).

   (b) Let λ_1 and λ_2 denote the two distinct values of λ obtained in part (a). Verify that the function $u\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{for } t \in \mathbb{R},$$

   where c_1 and c_2 are constant, solves the differential equation in (1).

5. We showed in class that the square of the area of the parallelogram, $\mathcal{P}(u, v)$, determined by vectors u and v in $\mathbb{R}^2$ satisfies the equation
$$(\text{area}(\mathcal{P}(u, v)))^2 = \|u\|^2\|v\|^2 - (v \cdot u)^2. \tag{3}$$
   (a) Use the expression in (3) and properties of the dot product to derive the expression
$$\text{area}(\mathcal{P}(u, v))) = \|u\|\|v\||\sin\theta|, \tag{4}$$
   where θ is the angle between u and v.

   (b) Give a geometric explanation of the expression in (4).

   (c) When is the area of the parallelogram determined by u and v the largest possible?

6. Let A and Q denote the 2×2 matrices $A = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix}$

   (a) Show that Q is invertible, and compute its inverse, Q^{-1}.

   (b) Compute $Q^{-1}AQ$. Explain why $Q^{-1}AQ$ is called a diagonal matrix.

7. The matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1 and λ_2 are real numbers, is called a **diagonal** matrix.

   (a) Compute D^2, D^3 and D^n, for any positive integer n.

   (b) Assume that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Show that D is invertible and compute D^{-1}.

8. Consider the linear system
$$\begin{cases} \dot{x} &= -3x + 2y; \\ \dot{y} &= 4x - 5y. \end{cases} \tag{5}$$
   Let
$$\text{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \text{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
   and define the vector value function
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-7t}\text{v}_1 + c_2 e^{-t}\text{v}_2, \quad \text{for } t \in \mathbb{R}, \tag{6}$$
   where c_1 and c_2 are constants.

(a) Verify that the vector–valued function given in (6) solves the system in (5).

(b) Use (6) to sketch trajectories of the system in (5) for the cases

(i) $c_1 = 0$ and $c_2 = 0$;

(ii) $c_1 \neq 0$ and $c_2 = 0$;

(iii) $c_1 = 0$ and $c_2 \neq 0$.

9. Consider the Lotka–Volterra system

$$\begin{cases} \dot{x} = x - xy; \\ \dot{y} = xy - y. \end{cases} \tag{7}$$

Use the Chain Rule to derive

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

and use this expressions to obtain an equation satisfied by the trajectories of the system in (7) for $x > 0$ and $y > 0$.

10. Let a, b, c and d denote real numbers, and consider the system of linear equations

$$\begin{cases} ax + by = 0; \\ cx + dy = 0. \end{cases} \tag{8}$$

(a) Explain why $x = y = 0$ solves the system in (8). This solution is usually referred to as the trivial solution of the system in (8).

(b) Show that, if $ad - bc \neq 0$, then the system in (8) has only the trivial solution.

(c) Assume that $ad - bc = 0$ and $a \neq 0$. Compute all the solutions of the system in (8) in this case.