Solutions to Review Problems for Exam 2

1. Compute and sketch the flow of the vector field

$$F(x,y) = -2x\hat{\imath} + y\hat{\jmath}, \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

Solution: First, we compute solutions of the system of differential equations

$$\begin{cases} \dot{x} &= -2x; \\ \dot{y} &= y. \end{cases} \tag{1}$$

The solution curves of the system in (1) are given parametrically by

$$\begin{cases} x(t) &= c_1 2^{-2t}; \\ y(t) &= c_2 e^t, \end{cases} \qquad \text{for } t \in \mathbb{R}, \tag{2}$$

and for constants of integration c_1 and c_2.

We sketch the various types of curves prametrized by the equations in (2) by considering all possibilities for c_1 and c_2.

If $c_1 = 0$ and $c_2 = 0$ in (2), we obtain the equilibrium solution $(0,0)$; this is sketched as a dot in Figure 1.

If $c_1 \neq 0$ and $c_2 = 0$ in (2), we obtain the parametric equations

$$\begin{cases} x(t) &= c_1 2^{-2t}; \\ y(t) &= 0, \end{cases} \qquad \text{for } t \in \mathbb{R},$$

which are the parametric equations of half–lines along the x–axis: the positive x–axis for $c_1 > 0$, and the negative x–axis for $c_1 < 0$. These trajectories tend towards the origin $(0,0)$ because e^{-2t} decreases to 0 as t increases. These trajectories are sketched in Figure 1.

If $c_1 = 0$ and $c_2 \neq 0$ in (2), we obtain the parametric equations

$$\begin{cases} x(t) &= 0; \\ y(t) &= c_2 e^t, \end{cases} \qquad \text{for } t \in \mathbb{R},$$

which are the parametric equations of half–lines along the y–axis: the positive y–axis for $c_2 > 0$, and the negative y–axis for $c_2 < 0$. These trajectories tend away from origin $(0,0)$ because e^t increases as t increases. These trajectories are sketched in Figure 1.

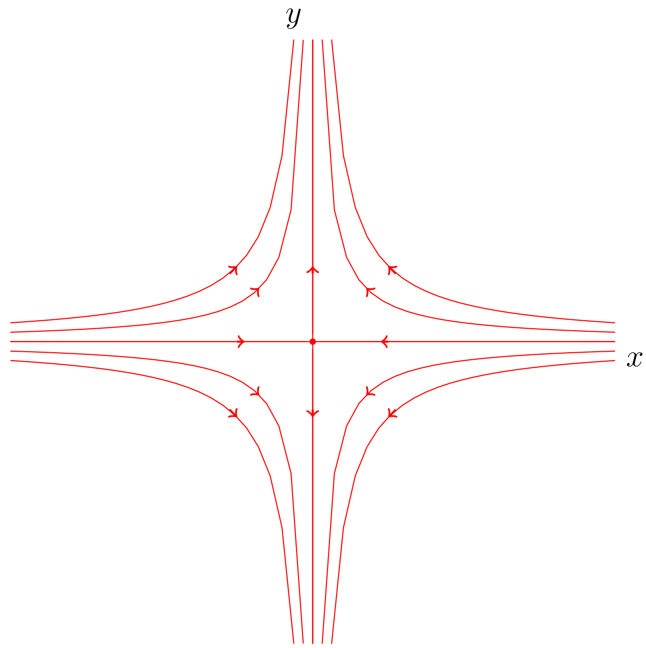

Figure 1: Sketch of Phase Portrait of System (1)

Finally, assume that $c_1 \neq 0$ and $c_2 \neq 0$. From the equations in (2) we obtain

$$\left\{ \begin{array}{rcl} x & = & c_1 2^{-2t}; \\ y^2 & = & c_2^2 e^{2t}, \end{array} \right. \qquad \text{for } t \in \mathbb{R}. \tag{3}$$

Multiplying the equations in (3) to each other then yields the equation

$$xy^2 = c_1 c_2^2,$$

or

$$xy^2 = c, \tag{4}$$

where we have set $c = c_1 c_2^2$; so that, $c \neq 0$.

The trajectories given by the equation in (4) lie on each of the four quadrants off the coordinate axis. For instance, for the case $c > 0$, we can solve (4) for y to obtain

$$y = \pm \frac{\sqrt{c}}{\sqrt{x}}, \qquad \text{for } x > 0.$$

These yield trajectories in the first and fourth quadrant in Figure 1. The trajectories in the second and third quadrant correspond to the case $c < 0$.

The directions along the trajectories given by (4) for $c \neq 0$ are dictated by the signs of $\dot{x}$ and $\dot{y}$ in each of the quadrants. These directions are shown by arrows of the curves shown in Figure 1. $\square$

2. Compute and sketch the flow of the vector field

$$F(x, y) = -2x\hat{\imath} - 2y\hat{\jmath}, \quad \text{for } (x, y) \in \mathbb{R}^2.$$

***Solution**:* First, we compute solutions of the system of differential equations

$$\begin{cases} \dot{x} = -2x; \\ \dot{y} = -2y. \end{cases} \tag{5}$$

The solution curves of the system in (5) are given parametrically by

$$\begin{cases} x(t) = c_1 2^{-2t}; \\ y(t) = c_2 e^{-2t}, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{6}$$

and for constants of integration c_1 and c_2.

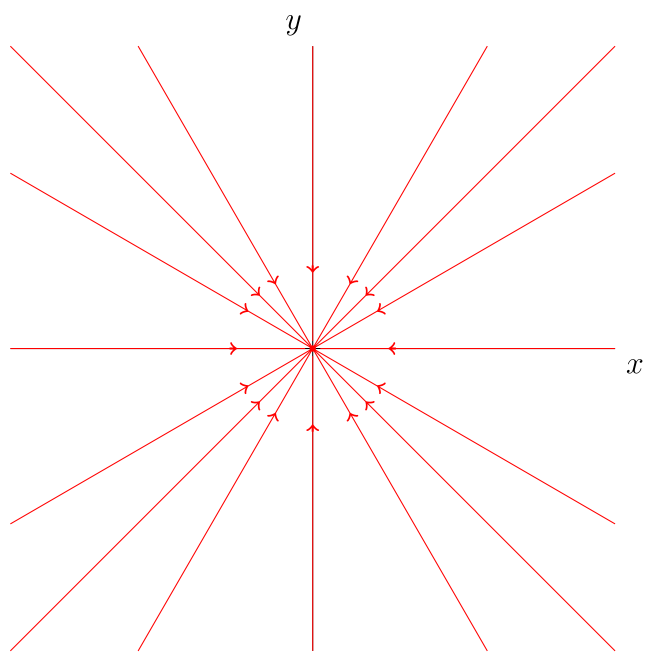

Figure 2: Sketch of Phase Portrait of System (5)

The case $c_1 = c_2 = 0$ in (6) corresponds to the equilibrium point $(0, 0)$, which is sketched as dot in Figure 2.

The case $c_1 \neq 0$ and $c_2 = 0$ in (6) corresponds to two trajectories along the x–axis: one on the positive x–axis for $c_1.0$, and the other on the negative x–axis for $c_1 < 0$. Since e^{-2t} decreases to 0 as t increases, these two trajectories point toward the origin. These are shown in Figure 2.

The case $c_1 = 0$ and $c_2 \neq 0$ in (6) corresponds to two trajectories along the y–axis: one on the positive y–axis ($c_2 > 0$) tending towards the origin since e^{-2t} decreases to 0 as t increases, and the other on the negative y–axis ($c_2 < 0$) also tending towards the origin. These two trajectories are sketched in Figure 2.

Finally, in the case $c_1 \neq 0$ and $c_2 \neq 0$ in (6), divide the first equation in (6) into the second equation to get

$$\frac{y}{x} = \frac{c_2}{c_1},$$

or

$$\frac{y}{x} = c,$$

where we have set $c = \dfrac{c_2}{c_1}$; so that,

$$y = cx, \tag{7}$$

where $c \neq 0$. Thus, the rest of the trajectories of the system in (5) lie along straight lines through the origin of non–zero slope. These trajectories all tend towards the origin since $e^{-2t} \to 0$ as $t \to \infty$. A few of those trajectories are shown in Figure 2. □

3. A particle of unit mass is moving along a path in the xy–plane parametrized by $\sigma(t) = R\sin(\omega t)\hat{i} + R\cos(\omega t)\hat{j}$, for $t \in \mathbb{R}$, where R is measured in meters, t is measured in seconds, and ω in radians per second.

   The particle flies of its path on a tangent line at time t_o such that $\omega t_o = \dfrac{\pi}{3}$ radians.

   (a) Give the position and velocity of the particle at time t_o.

   ***Solution**:* At time t_o the particle flies of its original path along a straight line parametrized by

$$\ell(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o), \qquad \text{for } t \geqslant t_o, \tag{8}$$

where

$$\sigma(t) = R\sin(\omega t)\hat{\imath} + R\cos(\omega t)\hat{\jmath}, \qquad \text{for } t \in \mathbb{R}, \tag{9}$$

and

$$\sigma'(t) = R\omega\cos(\omega t)\hat{\imath} - R\omega\sin(\omega t)\hat{\jmath}, \qquad \text{for } t \in \mathbb{R}. \tag{10}$$

The position of the particle at time t_o is obtained by substituting $\omega t_o = \dfrac{\pi}{3}$ in (9) to get

$$\sigma(t_o) = R\frac{\sqrt{3}}{2}\hat{\imath} + \frac{R}{2}\hat{\jmath}. \tag{11}$$

The velocity of the particle at time t_o is obtained by substituting $\omega t_o = \dfrac{\pi}{3}$ in (10) to get

$$\sigma'(t_o) = \frac{R\omega}{2}\hat{\imath} - \frac{R\omega\sqrt{3}}{2}\hat{\jmath}. \tag{12}$$

□

(b) Give the equation of the path of the particle after it flies off its circular path.

***Solution**:* Substitute the vectors in (11) and (12) into the expression for the tangent line to the path σ at t_o given in (8) to get

$$\ell(t) = R\frac{\sqrt{3}}{2}\hat{\imath} + \frac{R}{2}\hat{\jmath} + (t - t_o)\left(\frac{R\omega}{2}\hat{\imath} - \frac{R\omega\sqrt{3}}{2}\hat{\jmath}\right), \qquad \text{for } t \geqslant t_o,$$

or

$$\ell(t) = \left(R\frac{\sqrt{3}}{2} + (t - t_o)\frac{R\omega}{2}\right)\hat{\imath} + \left(\frac{R}{2} - (t - t_o)\frac{R\omega\sqrt{3}}{2}\right)\hat{\jmath}, \tag{13}$$

for $t \geqslant t_o$. □

(c) Find the time $t > t_o$, if any, at which the particle meets the x–axis. Give the location of the particle at that time.

***Solution**:* The tangent line in (13) will meet the x–axis when the second component in (13 is 0, or

$$\frac{R}{2} - (t - t_o)\frac{R\omega\sqrt{3}}{2} = 0,$$

or

$$1 - (t - t_o)\omega\sqrt{3} = 0. \tag{14}$$

Solving (14) for t then yields

$$t = t_o + \frac{\sqrt{3}}{3\omega}.$$

□

4. A particle moving in a straight line (along the x–axis) is moving according to the law of motion

$$\ddot{x} = 8x - 2\dot{x}. \tag{15}$$

Define

$$x(t) = e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{16}$$

(a) Determine distinct values of λ for which the function x defined in (16) solves the differential equation in (15).

***Solution**:* Differentiate the function x in (16) with respect to t twice to get

$$\dot{x}(t) = \lambda e^{\lambda t}, \quad \text{for } t \in \mathbb{R}, \tag{17}$$

and

$$\ddot{x}(t) = \lambda^2 e^{\lambda t}, \quad \text{for } t \in \mathbb{R}, \tag{18}$$

where we have used the Chain Rule.

Substituting the expressions for x, $\dot{x}$ and $\ddot{x}$ in (16), (17) and (18), respectively, into the differential equation in (15) yields

$$\lambda^2 e^{\lambda t} = 8e^{\lambda t} - 2\lambda e^{\lambda t}, \quad \text{for } t \in \mathbb{R},$$

from which we get

$$\lambda^2 = 8 - 2\lambda,$$

since the exponential function is never 0; from which we get the second–order equation

$$\lambda^2 + 2\lambda - 8 = 0. \tag{19}$$

The left–hand side of (19) can be factored to yield

$$(\lambda + 4)(\lambda - 2) = 0,$$

from which we get that

$$\lambda_1 = -4 \quad \text{and} \quad \lambda_2 = 2. \tag{20}$$

□

(b) Let λ_1 and λ_2 denote the two distinct values of λ obtained in part (a). Verify that the function $u\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{for } t \in \mathbb{R}, \tag{21}$$

where c_1 and c_2 are constants, solves the differential equation in (15).

***Solution**:* With the values of λ_1 and λ_2 given in (20), we obtain from (21) that

$$u(t) = c_1 e^{-4t} + c_2 e^{2t}, \quad \text{for } t \in \mathbb{R}, \tag{22}$$

where c_1 and c_2 are constants.

Differentiating the function u in (22) with respect to t twice then yields

$$\dot{u}(t) = -4c_1 e^{-4t} + 2c_2 e^{2t}, \quad \text{for } t \in \mathbb{R}, \tag{23}$$

and

$$\ddot{u}(t) = 16c_1 e^{-4t} + 4c_2 e^{2t}, \quad \text{for } t \in \mathbb{R}. \tag{24}$$

Next, compute

$$\begin{aligned} 8u(t) - 2\dot{u}(t) &= 8(c_1 e^{-4t} + c_2 e^{2t}) - 2(-4c_1 e^{-4t} + 2c_2 e^{2t}) \\ &= 8c_1 e^{-4t} + 8c_2 e^{2t} + 8c_1 e^{-4t} - 4c_2 e^{2t}, \end{aligned}$$

from which we get that

$$8u(t) - 2\dot{u}(t) = 16c_1 e^{-4t} + 4c_2 e^{2t}, \quad \text{for } t \in \mathbb{R}. \tag{25}$$

Comparing (24) and (25), we see that

$$\ddot{u}(t) = 8u(t) - 2\dot{u}(t), \quad \text{for } t \in \mathbb{R},$$

which shows that the function u in (22) solves the differential equation in (15). □

5. We showed in class that the square of the area of the parallelogram, $\mathcal{P}(u, v)$, determined by vectors u and v in $\mathbb{R}^2$ satisfies the equation

$$(\text{area}(\mathcal{P}(u, v)))^2 = \|u\|^2\|v\|^2 - (v \cdot u)^2. \tag{26}$$

(a) Use the expression in (26) and properties of the dot product to derive the expression

$$\text{area}(\mathcal{P}(u,v))) = \|u\|\|v\||\sin\theta|, \tag{27}$$

where θ is the angle between u and v.

***Solution**:* Use the fact that $v \cdot u = \|v\|\|u\|\cos\theta$ to get from (26) that

$$\begin{aligned}(\text{area}(\mathcal{P}(u,v)))^2 &= \|u\|^2\|v\|^2 - (\|v\|\|u\|\cos\theta)^2 \\ &= \|u\|^2\|v\|^2 - \|v\|^2\|u\|^2\cos^2\theta \\ &= \|u\|^2\|v\|^2(1-\cos^2\theta),\end{aligned}$$

from which we get that

$$(\text{area}(\mathcal{P}(u,v)))^2 = \|u\|^2\|v\|^2\sin^2\theta. \tag{28}$$

Taking the positive square root on both sides of (28 yields (27). □

(b) Give a geometric explanation of the expression in (27).

***Solution**:* Refer to Figure 3.

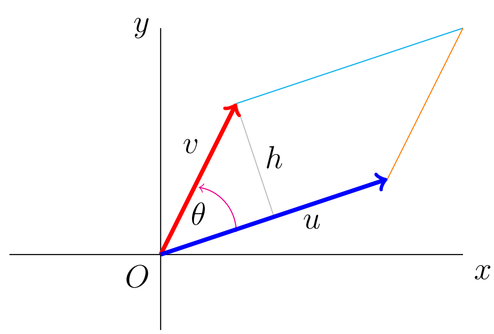

Figure 3: Parallelogram determined by u and v

The sketch in Figure 3 shows vectors u and v in standard position in the first quadrant. The sketch also shows the parallelogram, $\mathcal{P}(u,v)$, determined by u and v. The sketch also shows that angle, θ, between and b, and the height, h, of the parallelogram (the distance from v to the line through O in the direction of u.

The area of the parallelogram in Figure 3 is given by

$$\text{area}(\mathcal{P}(u,v))) = \|u\|h, \tag{29}$$

the are of the base times the height.

The line determining the height is perpendicular to the line through O in the direction of u. Hence v is the hypotenuse of a right triangle determined by height line, u and v. Hence, by the definition of the sine function,

$$\sin\theta = \frac{h}{\|v\|},$$

from which we get

$$h = \|v\| \sin\theta. \tag{30}$$

Substituting the expression for h in (30) into (29) yields (27). □

(c) When is the area of the parallelogram determined by u and v the largest possible?

***Solution**:* It follows from (27) that area($\mathcal{P}(u,v)$)) is the largest when $|\sin\theta| = 1$. This occurs when θ is a right angle. Hence, the parallelogram must be a rectangle for its area to be the largest possible. □

6. Let A and Q denote the 2×2 matrices $A = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix}$

(a) Show that Q is invertible, and compute its inverse, Q^{-1}.

***Solution**:* Compute $\det(Q) = 6 \neq 0$. Consequently, Q is invertible and its inverse is given by

$$Q^{-1} = \frac{1}{6}\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}. \tag{31}$$

□

(b) Compute $Q^{-1}AQ$. Explain why $Q^{-1}AQ$ is called a diagonal matrix.

***Solution**:* Use the associative property of matrix multiplication to compute

$$\begin{aligned} Q^{-1}AQ &= \frac{1}{6}\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -4 & 2 \\ 16 & 4 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} -24 & 0 \\ 0 & 12 \end{pmatrix}; \end{aligned}$$

so that,

$$Q^{-1}AQ = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}.$$

This matrix is diagonal because the nonzero entries are along the main diagonal of the matrix. □

7. The matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1 and λ_2 are real numbers, is called a **diagonal** matrix.

   (a) Compute D^2, D^3 and D^n, for any positive integer n.
   ***Solution**:* Compute

$$\begin{aligned} D^2 &= DD \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}. \end{aligned}$$

   Next, use the associative property of matrix multiplication to compute

$$\begin{aligned} D^3 &= DD^2 \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}. \end{aligned}$$

   The calculations shown above suggest that

$$D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix},$$

   for positive integers n. □

   (b) Assume that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Show that D is invertible and compute D^{-1}.

***Solution**:* In this case, $\det(D) = \lambda_1\lambda_2 \neq 0$; so that, D is invertible and

$$D^{-1} = \frac{1}{\lambda_1\lambda_2}\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

or

$$D^{-1} = \begin{pmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{pmatrix},$$

or

$$D^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}.$$

□

8. Consider the linear system

$$\begin{cases} \dot{x} = -3x + 2y; \\ \dot{y} = 4x - 5y. \end{cases} \tag{32}$$

Let

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and define the vector value function

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-7t}\mathrm{v}_1 + c_2 e^{-t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R}, \tag{33}$$

where c_1 and c_2 are constants.

(a) Verify that the vector–valued function given in (33) solves the system in (32).

***Solution**:* Write the system in (33) in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{34}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} -3 & 2 \\ 4 & -5 \end{pmatrix}. \tag{35}$$

Observe that

$$A\mathrm{v}_1 = \begin{pmatrix} -3 & 2 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 14 \end{pmatrix} = -7\begin{pmatrix} 1 \\ -2 \end{pmatrix};$$

so that,

$$A\mathrm{v}_1 = -7\mathrm{v}_1. \tag{36}$$

Similarly,

$$A\mathrm{v}_2 = \begin{pmatrix} -3 & 2 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix};$$

so that,

$$A\mathrm{v}_2 = -\mathrm{v}_2. \tag{37}$$

Taking the derivative with respect to t of the vector valued function in (33), we obtain

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c_1(-7)e^{-7t}\mathrm{v}_1 + c_2(-1)e^{-t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R};$$

so that, using the associative property,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c_1e^{-7t}(-7\mathrm{v}_1) + c_2e^{-t}(-\mathrm{v}_2), \quad \text{for } t \in \mathbb{R}.$$

Hence, in view of (36) and (37),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c_1e^{-7t}A\mathrm{v}_1 + c_2e^{-t}A\mathrm{v}_2, \quad \text{for } t \in \mathbb{R};$$

so that, using the distributive property of matrix multiplication

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A(c_1e^{-7t}\mathrm{v}_1 + c_2e^{-t}\mathrm{v}_)), \quad \text{for } t \in \mathbb{R}. \tag{38}$$

Comparing (33) and (38), we see that

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

which shows that the vector–valued function in (33) solves the equation in (34), where A is given in (35). The differential equation in (34) is equivalent to the system in (32). Therefore, the vector–valued function given in (33) solves the system in (32), which was to be shown. □

(b) Use (33) to sketch trajectories of the system in (32) for the cases

(i) $c_1 = 0$ and $c_2 = 0$;

(ii) $c_1 \neq 0$ and $c_2 = 0$;

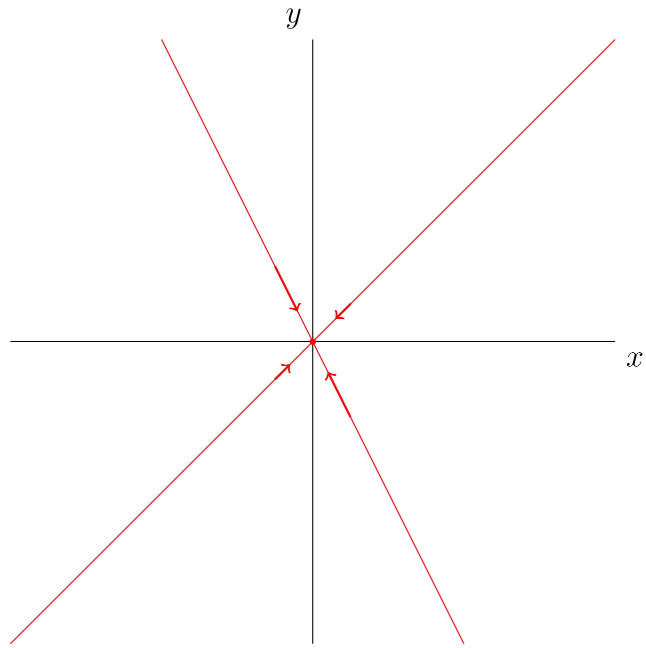

Figure 4: Sketch of solutions in (33) for cases (i), (ii) and (iii)

(iii) $c_1 = 0$ and $c_2 \neq 0$.

***Solution**:* Refer to the sketch in Figure 4.

(i) If $c_1 = c_2 = 0$ in (33),

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

which corresponds to the equilibrium solution $(0, 0)$; this is sketched as a dot in Figure 4.

(ii) If $c_2 = 0$ and $c_1 \neq 0$ in (33), then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-7t} \mathrm{v}_1, \qquad \text{for } t \in \mathbb{R}. \tag{39}$$

The equation in (39) is the vector–parametric equation of a half–line through the origin in the direction of v_1 if $c_1 > 0$, or a half–line through the origin through a direction opposite that of v_1 if $c_1 < 0$. Thus, there are two trajectories on the line parametrized by the equation in (39) that tend to $(0, 0)$ because e^{-7t} decreases to 0 as t increases. These trajectories are shown in the sketch in Figure 4.

(iii) If $c_1 = 0$ and $c_2 \neq 0$, (33) yields the vector–parametric equation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_2 e^{-t} \text{v}_2, \quad \text{for } t \in \mathbb{R}. \tag{40}$$

The equation in (40) is a parametrization of two trajectories of the system in (32): a half–line through the origin in the direction of the vector v_2 corresponding to the case $c_2 > 0$, and a half–line in the opposite direction corresponding to the case $c_2 < 0$. Both trajectories tend towards the origin because e^{-t} decreases to 0 as t increases.

□

9. Consider the Lotka–Volterra system

$$\begin{cases} \dot{x} = x - xy; \\ \dot{y} = xy - y. \end{cases} \tag{41}$$

Use the Chain Rule to derive

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \tag{42}$$

and use this expressions to obtain an equation satisfied by the trajectories of the system in (41) for $x > 0$ and $y > 0$.

***Solution**:* Use the expression in (42) to obtain the differential equation

$$\frac{dy}{dx} = \frac{xy - y}{x - xy},$$

or

$$\frac{dy}{dx} = \frac{(x-1)y}{x(1-y)}. \tag{43}$$

The differential equation in (43) can be separated to yield

$$\frac{1-y}{y} dy = \frac{x-1}{x} dx,$$

or

$$\left(\frac{1}{y} - 1\right) dy = \left(1 - \frac{1}{x}\right) dx. \tag{44}$$

Integrating on both sides of (44)

$$\int \left(\frac{1}{y} - 1\right) dy = \int \left(1 - \frac{1}{x}\right) dx,$$

yields

$$\ln|y| - y = x - \ln|x| + C, \tag{45}$$

where C is a constant of integration.

The expression in (45) is an equation satisfied by the trajectories of the system in (41). □

10. Let a, b, c and d denote real numbers, and consider the system of linear equations

$$\begin{cases} ax + by &= 0; \\ cx + dy &= 0. \end{cases} \tag{46}$$

(a) Explain why $x = y = 0$ solves the system in (46). This solution is usually referred to as the trivial solution of the system in (46).

***Solution**:* Substituting 0 for x and 0 for y in the left–hand side of the equations in (46) yields 0 in the left–hand sides of the equations. Thus, the equations are satisfies simultaneously in this case. □

(b) Show that, if $ad - bc \neq 0$, then the system in (46) has only the trivial solution.

***Solution**:* The system in (46) can be written in matrix form

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{47}$$

where A is the 2×2 matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{48}$$

Since $\det(A) = ad - bc \neq 0$, the matrix A in (47) has an inverse A^{-1}. Multiply on both sides of the equation in (47) by A on the left to get

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

so that, using the associative property of matrix multiplication,

$$(A^{-1}A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{49}$$

where I is the 2×2 identity matrix.

It follows from (49), and the calculations leading to it, that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the only solution of the equation in (47), which is equivalent to the system in (46). □

(c) Assume that $ad - bc = 0$ and $a \neq 0$. Compute all the solutions of the system in (46) in this case.

***Solution**:* Assume that $ad - bc = 0$ and $a \neq 0$.

Then,

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 0;$$

so that, the parallelogram determined by the vectors $\begin{pmatrix} a & b \end{pmatrix}$ and $\begin{pmatrix} c & d \end{pmatrix}$ has zero area. Consequently, the vector $\begin{pmatrix} a & b \end{pmatrix}$ lies in the same line as the vector $\begin{pmatrix} c & d \end{pmatrix}$. Therefore, $\begin{pmatrix} a & b \end{pmatrix}$ is a scalar multiple of $\begin{pmatrix} c & d \end{pmatrix}$; so that

$$\begin{pmatrix} a & b \end{pmatrix} = \lambda \begin{pmatrix} c & d \end{pmatrix}. \tag{50}$$

It follows from (50) that $\lambda \neq 0$, since we are assuming that $a \neq 0$.

Multiply the second equation in (46) by λ to get

$$\begin{cases} ax + by &= 0; \\ \lambda cx + \lambda dy &= 0, \end{cases}$$

which, in view of (50), is equivalent to

$$\begin{cases} ax + by &= 0; \\ ax + by &= 0, \end{cases}$$

Hence, the system in (46) reduces to the single equation

$$ax + by = 0. \tag{51}$$

Thus, all points on the line in (51) solve the system (46).

Solving the equation in (51) for x yields

$$x = -\frac{b}{a} y \tag{52}$$

Thus, setting $y = -at$, where t is a parameter, we obtain the parametric equations

$$\begin{cases} x = bt; \\ y = -at. \end{cases}$$

Thus, the solutions of the system in (46) are all the scalar multiples of the vector $\begin{pmatrix} b \\ -a \end{pmatrix}$. $\square$