Assignment #3

Due on Friday, February 14, 2020

Read Section 2.4, *Example: Modeling the Spread of an Infectious Disease,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

In Section 2.4 of the online class notes, the basic epidemiology model

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta SI}{N}; \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{N} - \gamma I; \\ \dfrac{dR}{dt} = \gamma I, \end{cases} \tag{1}$$

of Kermack and McKendrick is discussed. The system in (1) is also an example of an SIR model. The quantity $S(t)$ denotes the number of individuals in a population of size N that are susceptible to acquiring a disease by coming into contact with infected individuals at time t; $I(t)$ is the number of individuals in the population that have the disease and and can infect susceptible individuals at time t; $R(t)$ is the number of individuals in the population that have recovered from the disease at time t and can no longer be infected. It is assumed that

$$S(t) + I(t) + R(t) = N, \quad \text{for all } t \geqslant 0,$$

where N is a constant parameter.

Do the following problems

1. Give interpretations for the parameters β and γ in the SIR model in (1). In particular, give the units of β and γ.

2. Assume that at time $t = 0$ all individuals in the population have the disease. Compute $I(t)$ for all $t \geqslant 0$ and $R(t)$ for all $t \geqslant 0$.

3. Explain how the SIR system in (1) can be transformed to the dimensionless system
$$\begin{cases} \dfrac{ds}{d\tau} &= -R_o si; \\ \\ \dfrac{di}{d\tau} &= R_o si - i; \\ \\ \dfrac{dr}{d\tau} &= i. \end{cases} \tag{2}$$
Give an expression for R_o and provide an interpretation.

4. Explain why it suffices to analyze the two–dimensional system
$$\begin{cases} \dfrac{ds}{d\tau} &= -R_o si; \\ \\ \dfrac{di}{d\tau} &= R_o si - i, \end{cases} \tag{3}$$
to understand the three–dimensional system in (2).

   Use the second equation in (3) to deduce that, if $R_o > 1$, then $\dfrac{di}{d\tau} > 0$ when s is very close to 1. Explain why, in this case, an outbreak of the disease will occur.

5. Use the chain rule to show that the two–dimensional system in (3) implies the following first–order ODE
$$\frac{di}{ds} = \frac{1}{R_o s} - 1. \tag{4}$$
Use separation of variables to solve the ordinary differential equation in (4) and give a formula for i as a function of s.