Assignment #8

Due on Monday, April 13, 2020

Read Section 4.1.5 on *The Poisson Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.1.6 on *Estimating Mutation Rates in Bacterial Populations* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.2 on *Random Processes* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems

1. *Poisson Process.* A collection of discrete random variable, $Y(t)$, for $t \geqslant 0$, with possible values $0, 1, 2, 3, \ldots$, is said to be a Poisson process with rate λ if the following conditions are satisfied:

   (i) $Y(0) = 0$.

   (ii) For $0 \leqslant t_1 < t_2 < t_3 < \cdots < t_n$, the random variables
   $$Y(t_2) - Y(t_1), Y(t_3) - Y(t_2), \ldots, Y(t_n) - Y(t_{n-1})$$
   are mutually independent. (Independent increments).

   (iii) For $0 \leqslant s < t$, the random variable $Y(t) - Y(s)$ has a Poisson distribution with parameter $\lambda(t - s)$; that is,
   $$\Pr(Y(t) - Y(s) = k) = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)}, \quad \text{for } k = 0, 1, 2, \ldots$$

   Assume the number of customers arriving at a grocery store can be modeled by a Poisson process with rate λ of 6 customers per hour.

   (a) Compute the probability that there at least 2 customers will arrive between 8:00 am 8:20 am.

   (b) Compute the probability that no costumers will come to the store between 8:00 am 8:20 am.

2. *Another Poisson Process Problem.* Assume the number, $M(t)$, of mutations in the time interval $[0, t]$ in a bacterial colony is a Poisson process with rate λ mutations per unit of time. Assume that in one unit of time, out of 87 colonies, 29 show no mutations. Use this information to estimate λ. Explain the reasoning leading to your answer.

3. *Modeling Survival Time after a Treatment.* Consider a group of people who have received a treatment for a disease such as cancer. Let T denote the *survival time;* that is, T is the number of years a person lives after receiving the treatment.

   Assume that the probability that a person receiving the treatment at time t will not survive past time $t+\Delta t$ is proportional to Δt; denote the constant of proportionality by $\mu > 0$. If we let $p(t)$ denote the probability that a person who received the treatment at time $t_o = 0$ is still alive at time t, obtain a differential equation for $p(t)$ and solve for $p(t)$ assuming that $p(0) = 1$.

4. *Modeling Survival Time after a Treatment, (Continued).* Let T, μ and $p(t)$ be as in Problem 3.

   (a) Explain why $\Pr(T > t) = p(t)$.

   (b) Give a formula for computing $F_{_T}(t) = \Pr(T \leqslant t)$, for all $t > 0$.
   $F_T(t)$, is called the *cumulative distribution function,* or cdf, of the random variable T.

   (c) Let $f_{_T}(t) = F'_{_T}(t)$ for all $t > 0$. Show that $f_{_T}$ is of the form

   $$f_T(t) = \begin{cases} \dfrac{1}{\beta} e^{-t/\beta}, & \text{for } t > 0; \\ 0, & \text{for } t \leqslant 0, \end{cases}$$

   for some positive constant β.
   What is β in terms of μ?

   (d) Find the expected value of T; that is, compute $E(T) = \displaystyle\int_{-\infty}^{\infty} t f_T(t)\, \mathrm{d}t$.

5. *Modeling Survival Time after a Treatment, (Continued).* Let T have the distribution found in Problem 4.

   Define the survival function, $S(t)$, to be the probability that a randomly selected person will survive for at least t years after receiving treatment.

   (a) Compute $S(t)$ for all $t > 0$.

   (b) Suppose that a patient has a 70% probability of surviving at least two years. Find β, where β is the parameter defined in Problem 4.