Homework 6: Compact Spaces

"A child[s] ... first geometrical discoveries are topological ... If you ask him to copy a square or a triangle, he draws a closed circle." –Jean Piaget

Note: Feel free to use results from Math 131 and results from Kosniowski, but you must state what you are using.

1. A space X is said to be *locally compact* if for every $p \in X$ there is an open set W containing p, such that $\overline{W}$ is compact. Prove that the product of two locally compact spaces is locally compact. Is the product of infinitely many locally compact spaces necessarily locally compact? Prove your assertion.

2. Let X be a compact metric space, and let $\omega = \{U_j | j \in J\}$ be an open cover of X. Prove that there exists an $r > 0$ such that for any $A \subseteq X$, if $\text{lub}\{d(p,q)|p,q \in A\} < r$ then $A \subseteq U_j$ for some $j \in J$.

3. Let A be a compact subset of a metric space X. Let $b \in X - A$. Define $d(A,b) = \text{glb}\{d(p,b)|p \in A\}$.
a) Prove that there exists a point $a \in A$ such that $d(A,b) = d(a,b)$.
b) Suppose that $X = \mathbb{R}^n$ with the usual topology, and A is closed but not necessarily compact. For $b \in \mathbb{X} - A$, does there still exist a point $a \in A$ such that $d(A,b) = d(a,b)$?

4. Consider the rationals $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the usual topology. Let $A = \{q \in \mathbb{Q} | 0 \leqslant q \leqslant 1\}$. Determine whether or not A is compact in $\mathbb{Q}$. Prove all claims.

5. Let X be a topological space and let Y be a compact space. Let $f : X \to Y$, and define the graph of f as $G = \{(x, f(x))|x \in X\}$. Prove that if G is closed as a subset of the product space $X \times Y$ then f is continuous.