Homework 9: Homotopic Maps and Multiplication of Paths

"Topology is a relatively new branch of mathematics. Those who took training in mathematics 30 years ago did not have the opportunity to take a course in topology at many schools. Others had the opportunity, but passed it by, thinking topology was one of those 'new fangled' things that was not here to stay. In that respect, it was like the automobile. "One who is introduced to topology through popular lectures and entertaining articles may get the impression that topology is recreational mathematics. If he were to take a course in topology, expecting it to consist of cutting out pretty figures and stretching rubber sheets, he would be in for a rude awakening. If he pursued the subject further, however, he might be delighted to find that it is rich in substance and beauty." - R. H. Bing

1. Let X be a topological space and let $x, y \in X$. Let $P(x, y)$ denote the set of path homotopy classes in X from x to y. Show that there is a bijection from $P(x, y)$ to $P(x, x)$ if and only if $P(x, y)$ is non-empty.

2. Let X be a topological space and let Y be the space that consists of the single point c.
a) Prove that if a space X is homotopy equivalent to Y then X is path connected.
b) Let $f : X \to Y$ and $g : Y \to X$ be continuous and let $F : X \times I \to X$ be a homotopy from $g \circ f$ to the identity map. Let $p = g(c)$, and suppose that for all $t \in I$ we have $F(p, t) = p$. Prove that if γ is a loop in X based at p then $\gamma \sim e_p$.

3. Prove that if a topological space X is connected and is homotopy equivalent to a topological space Y, then Y is also connected.

4. Let X be a topological space and let $a \in X$. Let $n > 2$ be a natural number, and let f and g be loops in X based at a. Define loops h and k by

$$h(s) = \begin{cases} f(ns) \text{ if } s \in [0, \frac{1}{n}] \\ g(\frac{ns}{n-1} - \frac{1}{n-1}) \text{ if } s \in [\frac{1}{n}, 1] \end{cases}$$

and

$$k(s) = \begin{cases} f(\frac{ns}{n-1}) \text{ if } s \in [0, \frac{n-1}{n}] \\ g(ns + 1 - n) \text{ if } s \in [\frac{n-1}{n}, 1] \end{cases}$$

Prove that $h \sim k$.

5. Let X be a space, and let $f : S^1 \to X$ be a continuous map. Show that f is homotopic to a constant map if and only if there is a continuous map $g : D^2 \to X$ such that $g|S^1 = f$.