Math 152 – Fall 2006
Professor Hardin
Exam #1

Name: ________________________

1. (a) MLE: $\hat{\theta} = \overline{X} = \frac{49}{16} = 3.0625$
   (b) $P(X = 0) = e^{-\theta}$, MLE: $e^{-3.0625} = 0.0467$
   $I(\theta) = \frac{1}{\theta}$ (show work)
   $\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{D} N(0, g'(\theta)^2/I(\theta) = \theta e^{-2\theta})$
   Putting $e^{-\theta}$ in the middle, and using the MLE for the estimate of the variance, we get
   98% CI for $e^{-\theta}$: $e^{-\overline{X}} \pm 2.33\sqrt{e^{-2\overline{X}}\overline{X}/n}$: $(-0.049, 0.142)$
   Interpretation: we're 98% confident that the true probability of the next observations being zero is between -0.049 and 0.142. Note that we know the probability can't be negative. So, we could say we're 98% confident that the true probability of the next observations being zero is less than 0.142.

2. (a) $P(X_1 > 5.48|\theta = 4) = P(Z > \frac{5.48-4}{.9}) = P(Z > 1.645) = 0.05$
   $P(X_1 + X_2 > k|\theta = 4) = P(Z > \frac{k-8}{1.27}) = 0.05 \rightarrow k = 10.09$

   (b) $P(X_1 > 5.48|\theta = 5) = P(Z > \frac{5.48-5}{.9}) = P(Z > 0.5333) = 0.2981$
   $P(X_1 + X_2 > 10.09|\theta = 5) = P(Z > \frac{10.09-10}{1.27}) = P(Z > 0.071) = 0.4721$
   Test 2 is more powerful, which makes sense because it is based on 2 data points (more data usually means more power.)

3. (a) MLE: $\hat{\theta} = \min X_i$
   (b) MOM: set $E[X] = \overline{X}, \tilde{\theta} = (2/3)\overline{X}$
   (c) MSE$(\hat{\theta}) = Var(\hat{\theta}) + (E[\hat{\theta}] - \theta)^2 = \theta^2/10$
   MSE$(\tilde{\theta}) = Var(\tilde{\theta}) + (E[\tilde{\theta}] - \theta)^2 = \theta^2/81$
   MOM has smaller MSE (as well as MOM being unbiased!), so we should use MOM here.

4. (a)

$$\begin{aligned}
f(\underline{x}) &= \beta^n \prod_{i=1}^{n} (1 - x_i)^{\beta - 1} \\
l(\beta) &= n \ln \beta + (\beta - 1) \sum_{i=1}^{n} \ln(1 - x_i) \\
\frac{\partial l(\beta)}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln(1 - x_i) = 0 \\
\frac{\partial^2 l(\beta)}{\partial \beta^2} &= \frac{-n}{\beta} < 0 \text{ max!}
\end{aligned}$$

Solving for β we get $\hat{\beta} = -n/(\sum_i \ln(1 - x_i)) = -n/(\ln \prod_i (1 - x_i)) = 5.625$.

(b)

$$\Lambda = \frac{L(\beta_0)}{L(\hat{\beta})} = \frac{5^n[\prod_i(1-x_i)]^4}{5.625^n[\prod_i(1-x_i)]^{4.625}} = 0.716$$
$$-2\ln\Lambda = 0.667$$

Because we know that if H_0 is true, $-2\ln\Lambda \sim \chi_1^2$, at $\alpha = 0.05$ the cutoff will be 3.841. That is, our critical region is:

$$C = \{\underline{x}| - 2\ln\Lambda > 3.841\}$$

We do not reject H_0. There is not sufficient evidence to say that β is greater than 5.

(c) p-value $= P(\Lambda \leq 0.716) = P(-2\ln\Lambda \geq 0.667) = P(\chi_1^2 \geq 0.667) = 0.414$. The large p-value again tells us not to reject the null hypothesis.

5. (a)

$$\begin{aligned} P(Y_1 \leq y) &= 1 - P(Y_1 > y) = 1 - [P(X_i > y)]^{10} \\ &= 1 - [\int_y^{\infty} e^{-(x-\theta)} dx]^{10} = 1 - [-e^{\theta}e^{-x}|_y^{\infty}]^{10} \\ &= 1 - e^{-10(y-\theta)} \\ f_{Y_1}(y) &= 10e^{-10(y-\theta)} \end{aligned}$$

(b)

$$\begin{aligned} P(\theta \leq Y_1 \leq \theta + c) &= [1 - e^{-10(\theta+c-\theta)}] - [1 - e^{-10(\theta-\theta)}] \\ &= 1 - e^{-10c} = 0.9 \\ -10c &= \ln(0.1) \\ c &= 0.23 \\ P(\theta \leq Y_1 \leq \theta + 0.23) &= 0.9 \\ P(Y_1 - 0.23 \leq \theta \leq Y_1) &= 0.9 \end{aligned}$$

$(Y_1 - 0.23, Y_1)$ is a 90% confidence interval for θ.