Math 152 – Fall 2006
Professor Hardin
Exam #2
(out of 85 points)

Name: ______________________

1. (a)

$$\begin{aligned} f(\underline{x};\theta) &= \theta^n e^{-\sum x_i \theta} \\ h(\theta) &= 1/\theta \\ k(\theta;\underline{x}) &\propto \theta^{n-1} e^{-\theta \sum x_i} \\ \Theta|\underline{X} &\sim \text{ gamma}(n, \textstyle\sum x_i) \\ E(\Theta|\underline{X}) &= 1/\overline{X} \end{aligned}$$

   (b)

$$\begin{aligned} \ln f(\underline{x};\theta) &= n \ln \theta - \sum x_i \theta \\ \frac{\partial l(\theta)}{\partial \theta} &= n/\theta - \sum x_i = 0 \\ \hat{\theta} &= 1/\overline{X} \end{aligned}$$

   (c) If $h(\theta) = 1/\theta$ note that the prior mean is $E[\Theta] = \infty$. The expected value is undefined, so we don't really have any prior information about θ. Also, we can't have a weighted estimate of the prior guess and the frequentist estimate.

2. (a)

$$\begin{aligned} \frac{L(\lambda_0)}{L(\lambda_1)} &= e^{-n(\lambda_0-\lambda_1)}(\lambda_0/\lambda_1)^{\sum x_i} \leq k \\ &\iff \sum x_i \geq c \end{aligned}$$

   Because we know $\sum X_i \sim Poisson(n\lambda)$, we can use $\sum X_i$ to define our critical region:

$$\begin{aligned} \alpha &= P(\textstyle\sum X_i \geq c | \lambda = 1, n\lambda = 10) = 0.049 \text{ when } c = 16 \\ C &= \{\underline{x} | \textstyle\sum x_i \geq 16\} \end{aligned}$$

   (b) If we set up 2 simple hypotheses: $\lambda_0 = 1$ and $\lambda_1 > \lambda_0$, we can use the Neyman-Pearson theorem to get a MP test. Notice that this test is MP for ANY $\lambda_1 > \lambda_0$. So, it is UMP for $H_0 : \lambda = 1, H_1 : \lambda > 1$. Also notice that the level of significance is (always) the maximum over all values in Ω_0. The maximum α will happen at $\lambda = 1$ when $\Omega_0 : \lambda \leq 1$. So, the above test is UMP at $\alpha = 0.049$ for $H_0 : \lambda \leq 1, H_1 : \lambda > 1$.

(c) $\alpha = 0.049$

(d) $\sum x_i = 15$, so we cannot reject H_0. We do not have enough evidence to claim that $\lambda > 1$.

3. (a)

$$f(\underline{x};\theta) = \theta^n (1-\theta)^{\sum x_i} = \theta^n exp\{\sum x_i \ln(1-theta)\}$$

Because we can see that the distribution clearly belongs to the exponential family of distributions, we know $Y_1 = \sum X_i$ is a complete sufficient statistic.

(b) $E[Y_2] = E[h(X_1)] = 1 \cdot P(X_1 = 0) + 0 \cdot P(X_1 \neq 0) = P(X_1 = 0) = \theta$.

(c)

$$\begin{aligned} Y_3 &= E(Y_2|\sum X_i = t) = P(X_1 = 0|\sum X_i = t) \\ &= \frac{P(X_1 = 0 \ \ \& \ \ \sum X_i = t)}{P(\sum X_i = t)} = \frac{P(X_1 = 0)P(\sum_{i=2}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} \\ &= \frac{\theta \cdot \binom{(n-1)-1}{t}\theta^{n-1-t}(1-\theta)^t}{\binom{n-1}{t}\theta^{n-t}(1-\theta)^t} = \frac{\binom{n-2}{t}}{\binom{n-1}{t}} \\ &= \frac{(n-2)!/t!(n-2-t)!}{(n-1)!/t!(n-1-t)!} = \frac{n-1-t}{n-1} \end{aligned}$$

Notice that we calculated the above probabilities using: $\sum_{i=2}^n X_i \sim$ Neg Bin(n-1,θ), and $\sum_{i=1}^n X_i \sim$ Neg Bin(n,θ).

(d) We know that THE unbiased estimator based only on the complete sufficient statistic is the MVUE by the Lehmann-Scheffe theorem. $Y_3 = (n-1-\sum X_i)/(n-1)$ is the MVUE.

4. (a)

$$\begin{aligned} \frac{L(2)}{L(3)} &= \frac{2x}{3x^2} = \frac{2}{3x} \leq \frac{3}{2} \\ &\iff x \geq \frac{4}{9} \\ C &= \{x|x \geq 4/9\} \end{aligned}$$

(b)

$$\begin{aligned} \alpha &= P(X \geq 4/9|\theta = 2) = \int_{4/9}^{1} 2x dx = x^2|_{4/9}^{1} = 65/81 = 0.802 \\ \beta &= P(X < 4/9|\theta = 3) = \int_{0}^{4/9} 3x^2 dx = x^3|_0^{4/9} = 64/729 = 0.087 \\ 2\alpha + 3\beta &= 1.867 \end{aligned}$$

(c) Because α is so large!