Professor Hardin
Math 152 – Fall 2006
warm-up # 13

Name: ________________________

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson(λ) pdf:

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Find the MVUE of $\theta = P(X = 0)$.

Notice that $Y = \sum_i X_i$ is still a complete sufficient statistic for λ (from notes 10/26/06.) Also notice that if $X_i \overset{\text{iid}}{\sim}$ Poisson (λ) then $\sum_{i=1}^{m} X_i \sim$ Poisson($m\lambda$).

Solution:

$$\text{Let } Y_2 = h(X_1) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} E[Y_2] &= E[h(X_1)] = 1 * P(X_1 = 0) + 0 * P(X_1 \neq 0) = P(X_1 = 0) = \theta = e^{-\lambda} \\ \text{So, } \varphi(Y) &= E[Y_2 | Y = y] \text{ is the MVUE!} \end{aligned}$$

But what is $\varphi(Y)$...?

$$\begin{aligned} \varphi(Y) = E[Y_2|Y = y] &= P(X_1 = 0 | Y = y) \\ &= P(X_1 = 0 | \sum_i X_i = y) \\ &= \frac{P(X_1 = 0 \ \& \ \sum_i X_i = y)}{P(\sum_i X_i = y)} \\ &= \frac{P(X_1 = 0 \ \& \ \sum_{i=2}^{n} X_i = y)}{P(\sum_i X_i = y)} \\ &= \frac{P(X_1 = 0) P(\sum_{i=2}^{n} X_i = y)}{P(\sum_i X_i = y)} \\ &= \frac{[e^{-\lambda}][((n-1)\lambda)^y e^{-(n-1)\lambda}/y!]}{[(n\lambda)^{y_1} e^{-n\lambda}/y!]} \\ &= \frac{(n-1)^y}{n^y} \qquad \text{is the MVUE.} \end{aligned}$$

Note, we know that $\varphi(Y)$ is unbiased for θ (see proof of 7.3.1), and we know that $\varphi(Y)$ is a function of the complete sufficient statistic. Therefore, $\varphi(Y) = \frac{(n-1)^Y}{n^Y}$ (where $Y = \sum_i X_i$) is the MVUE from the Lehmann-Scheffé theorem.