Professor Hardin
Math 152 – Fall 2006
warm-up # 14

Name: ________________________

Suppose that a random sample, $X_1, X_2, \dots, X_n$, is taken from a uniform distribution on the interval $[0, \theta]$, where the value of θ is unknown, and the following simple hypotheses are to be tested:

$$H_0 : \theta = 1$$
$$H_1 : \theta = 2$$

1. Show that there exists a test procedure for which $\alpha = 0$ and $\beta < 1$.
2. Among all test procedures for which $\alpha = 0$, find the one for which β is a minimum. (You don't have to be able to prove that your value is a minimum, but you should be able to argue it.)
3. Find the minimum value of β that can be attained among all test procedures for which $\alpha = 0$.

Solution:

1. $$C = \{\mathbf{x} : \max(x_i) > 1\}$$
   will be a critical region that never creates a type I error. That is, we only reject the null hypothesis if the maximum value is bigger than 1. And in the case where the maximum value is bigger than 1, we know that the alternative is true (thus, we haven't made a mistake.)
2. The above critical region is the one for which β is a minimum. That is, any other function of the data (say, the average) giving no type I error will happen when the above critical region happens. So, the critical region defined above is the most powerful one.
3. To find β we need the distribution of the maximum. From previous work, we know $Y = \max(X_i)$:
   $$\begin{aligned} f_Y(y) &= \frac{ny^{(n-1)}}{\theta^n} \\ \beta &= P(Y < 1 | \theta = 2) \\ &= \int_0^1 \frac{ny^{(n-1)}}{2^n} dy \\ &= \frac{y^n}{2^n}|_0^1 = \left(\frac{1}{2}\right)^n \end{aligned}$$
   Notice how the type II error goes down as your sample size goes up. (That is, power goes up for larger samples.)