Professor Hardin
Math 152 – Fall 2006
warm-up # 16

Name: ______________________

This isn't really a warm-up because I'm going to tell you all the answers...

Remember that it is possible to define the power function over all values of θ (not just Ω_1.)

$$\delta_C(\theta) = \text{ Power at } \theta = P(\text{rejecting } H_0|\theta) = P(\mathbf{X} \in C|\theta)$$

A critical region is **unbiased** if

$$\delta_C(\theta') \leq \delta_C(\theta'') \text{ for all } \theta' \in \Omega_0 \text{ and } \theta'' \in \Omega_1$$

Suppose that a random sample, $X_1, X_2, \ldots, X_n$, is taken from a Normal(μ, σ^2) distribution where σ^2 is known and μ is unknown. We are interested in testing:

$$H_0 : \mu = 47$$
$$H_1 : \mu \neq 47$$

After applying the Neyman-Pearson theorem and Definition 8.2.1, we realize that

- if $H_1 : \mu < 47$ then:

  $$C_1 = \{\mathbf{x}|\overline{x} < c_1\}$$

  is a UMP critical region.

- if $H_1 : \mu > 47$ then:

  $$C_2 = \{\mathbf{x}|\overline{x} > c_2\}$$

  is a UMP critical region.

Therefore, we cannot have a UMP test over all possible values of μ in the alternative, Ω_1.

Note that if we are given $\alpha = 0.01$, $\sigma^2 = 9$ and $n = 10$, we can find c_1 and c_2. First, we'll assume $H_1 : \mu < 47$, and we'll find c_1:

$$\begin{aligned}
\alpha = 0.01 &= P(\overline{X} < c_1|\mu = 47) \\
&= P(\frac{\overline{X} - 47}{3/\sqrt{10}} < \frac{c_1 - 47}{3/\sqrt{10}}) \\
-2.33 &= \frac{c_1 - 47}{3/\sqrt{10}} \\
c_1 &= 44.79
\end{aligned}$$

Using similar equations, we find $c_2 = 49.21$.

Now, we want to show that the critical region C_1 is not unbiased for the alternative $H_1 : \mu \neq 47$. Using the definition of unbiased, we look for μ values in the null and alternative that might contratict the definition of unbiased:

$$\begin{aligned}
\delta_{C_1}(\mu = 47) &= P(\overline{X} < 44.79 | \mu = 47) = 0.01 \text{ (as it should)} \\
\delta_{C_1}(\mu = 46) &= P(\overline{X} < 44.79 | \mu = 46) \\
&= P\left(\frac{\overline{X} - 46}{3/\sqrt{10}} < \frac{44.79 - 46}{3/\sqrt{10}}\right) \\
&= P(Z < -1.28) \\
&= 1 - 0.8997 = 0.1003 \text{ (note that the power is bigger for values in the alternative!)} \\
\delta_{C_1}(\mu = 48) &= P(\overline{X} < 44.79 | \mu = 48) \\
&= P\left(\frac{\overline{X} - 48}{3/\sqrt{10}} < \frac{44.79 - 48}{3/\sqrt{10}}\right) \\
&= P(Z < -3.3836) \\
&= 1 - 0.9996 = 0.0004 \text{ (note here, the power is smaller for values in the alternative!)}
\end{aligned}$$

We found two values of μ, one each from the null and alternative, such that:

$$\delta_{C_1}(\mu' = 47) \quad > \quad \delta_{C_1}(\mu'' = 48)$$

So, the critical region C_1 is not unbiased.

It turns out that the critical region:

$$C_3 = \{\mathbf{x} | \quad |\overline{x} - \mu_0| > c_3\}$$

is unbiased. Let's find c_3...

$$\begin{aligned}
\alpha = 0.01 &= P(|\overline{X} - \mu_0| > c_3 | \mu = 47) \\
&= P(|\frac{\overline{X} - 47}{3/\sqrt{10}}| > \frac{c_3}{3/\sqrt{10}}) \\
&= P(|Z| > \frac{c_3}{3/\sqrt{10}}) \\
0.005 &= P(Z > \frac{c_3}{3/\sqrt{10}}) \\
2.57 &= \frac{c_3}{3/\sqrt{10}} \\
c_3 &= 3.16
\end{aligned}$$

I argue that you can't find any $\mu'' \neq 47$ such that

$$\delta_{C_3}(\mu' = 47) \quad > \quad \delta_{C_3}(\mu'')$$