Professor Hardin
Math 152 – Fall 2006
warm-up # 18

Name: ________________________

Let θ denote the average number of defects per 100 feet of a certain type of magnetic tape. Suppose that the value of θ is unknown, and the prior distribution of θ is a gamma distribution with parameters $\alpha = 2$ and $\beta = 10$. When a 1200-foot roll of this tape is inspected, exactly four defects are found. Find a 98% posterior interval for θ.

Use the alternative gamma parameterization:

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}$$

Note that we've taken 12 observations and seen $\sum x_i = 4$.
From WU #17...

$$\begin{aligned}
f(\underline{x}|\theta) &= \theta^{\sum x_i} e^{-n\theta} / \prod x_i \\
h(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta\beta} \\
k(\theta|\underline{x}) &\propto \theta^{\sum x_i + \alpha - 1} e^{-\theta(n+\beta)} \\
\theta|\underline{x} &\sim \text{Gamma}(\textstyle\sum x_i + \alpha, n+\beta) \\
&\sim \text{Gamma}(6, 22)
\end{aligned}$$

Solution:
If we want to find an interval for θ, we need to use the posterior distribution. Note that θ is now a random variable, so we can think of the interval as representing a probability:

$$0.98 = P(a \leq \theta \leq b|\underline{x})$$

Because we know that $\theta \sim$ Gamma (6,22), we simply need to find a Gamma(6,22) curve and look up the 0.025 and 0.975 values such that:

$$\begin{aligned}
a : 0.025 &= P(X \leq a) \\
b : 0.975 &= P(X \leq b)
\end{aligned}$$

Where $X \sim$ Gamma (6,22). We might also call $a = \Gamma_{0.025,6,22}$ and $b = \Gamma_{0.975,6,22}$, where Γ is defined as above.

A 98% posterior interval for θ is (0.081, 0.596). There is a 98% chance that the true θ is between 0.081 defects per 100 feet and 0.596 defects per 100 feet.