Math 152 – Fall 2007
Professor Hardin
warm-up # 18

Name: ________________________

Let $X_1, X_2, \ldots, X_{10}$ be a random sample from an $\exp(\theta)$ distribution, $E[X] = \theta$. Consider the following hypotheses:

$$H_o : \theta \leq 2$$
$$H_1 : \theta > 2$$

Notice:

- If $X_1, X_2, \ldots, X_n \sim \exp(2)$ then $\sum X_i \sim \chi^2_{2n}$.
- The MLE for θ in an exponential distribution is $\overline{X}$.

1. Find the likelihood ratio test at a level of $\alpha = 0.05$. (Hint: find $\ln(\Lambda)$ and look at it as a function of $\sum X_i$ or $\overline{X}$.)
2. Let's say you collect some data and find $\sum X_i = 29$, what is your conclusion?
3. What type of error might you have made?

Solution

1. $\delta : \{\text{reject } H_o \text{ if } \frac{\sup L_1}{\sup L_0} \geq k\}$

$$\begin{aligned}
\frac{\sup L_1}{\sup L_0} &= \frac{(\frac{1}{\theta_o})^n e^{-\sum X_i/\theta_o}}{(\frac{1}{\overline{X}})^n e^{-\sum X_i/\overline{X}}} \\
&= \left(\frac{\overline{X}}{2}\right)^n e^{-\sum X_i/2+n} \\
&\geq k \\
\Leftrightarrow \quad -n\ln(\overline{X}) + n\ln(2) + \frac{\sum X_i}{2} - n &\geq k \\
\Leftrightarrow \quad -n\ln(\overline{X}) + \frac{\sum X_i}{2} &\geq k \\
\Leftrightarrow \quad -n\ln(\sum X_i) + \frac{\sum X_i}{2} &\geq k
\end{aligned}$$

$$\text{Let } g(y) = -n\ln(y) + \frac{y}{2}.$$
$$\text{Then } \frac{\partial g(y)}{\partial y} = -\frac{n}{y} + \frac{1}{2}.$$
$$\text{And } \frac{\partial^2 g(y)}{\partial y^2} = \frac{n}{y^2} > 0.$$

So, we know that g(y) is minimized at $y = 2n$. However, we also know that $\sum X_i > 2n$ (because it makes sense that the MLE is in the alternative parameter space). So, if we want $g(\sum X_i)$ to be greater than some value, we need $\sum X_i$ to be bigger than some other value.

$$\delta : \{\text{reject } H_o \text{ if } \sum X_i \geq c\}$$
$$P(\sum X_i \geq c|\theta = 2) = P(\chi^2_{20} \geq c) = 0.05, c = 31.41.$$
$$\delta : \{\text{reject } H_o \text{ if } \sum X_i \geq 31.41\}$$

2. $\sum X_i = 29 \rightarrow$ do not reject H_o.

3. It's possible that we've made a type II error (that is, we didn't reject H_o, and it's possible that we should have.) It is **not** possible that we made a type I error.