Math 152 – Fall 2006
Professor Hardin
warm-up # 3

Name: _____

Let $Y_1 < Y_2 < \ldots < Y_5$ be the order statistics associated with 5 (n) independent observations $X_1, X_2, \ldots, X_5$, each with pdf

$$\begin{aligned} f_X(x) &= 3x^2 \qquad 0 < x < 1 \\ F_X(x) &= x^3 \qquad 0 < x < 1 \end{aligned}$$

Derive the cdf and pdf of Y_2, that is, the second biggest X_i.

Soultion

$$\begin{aligned} F_{Y_2}(y) &= P(Y_2 \leq y) = 1 - P(Y_2 > y) = 1 - P(4X_i > y \text{ and } 1X_i \leq y \text{ or all } 5X_i > y) \\ &= 1 - [P(4X_i > y \text{ and } 1X_i \leq y) + P(5X_i > y)] \\ &= 1 - [\binom{5}{4} P(X_i > y)^4 P(X_i \leq y) + P(X_i > y)^5] \\ &= 1 - [\binom{5}{4} (1 - F_X(y))^4 F_X(y) + (1 - F_X(y))^5] = 1 - [5 * (1-y^3)^4 y^3 + (1-y^3)^5] \\ &= 1 - [(1-y^3)^4 * (5y^3 + 1 - y^3)] \\ &= 1 - [(1-y^3)^4 * (4y^3 + 1)] \qquad 0 < y < 1 \end{aligned}$$

$$\begin{aligned} f_{Y_2}(y) &= -[(1-y^3)^4 12y^2 + (4y^3+1)4(1-y^3)^3(-3y^2)] = (1-y^3)^3 12y^2(4y^3 + 1 - 1 + y^3) \\ &= 60(1-y^3)^3 y^5 \quad 0 < y < 1 \end{aligned}$$