Math 152 – Fall 2007
Professor Hardin
warm-up # 5

Name: ________________________________

1. Show that $\hat{\theta} = \frac{X+1}{n+2}$ is a biased estimator of the binomial parameter θ (in a random sample of size n.)

2. Is $\hat{\theta}$ consistent for θ?

3. Find the Mean Square Error (MSE) for $\hat{\theta}$.
   (Aside for later: consider $\tilde{\theta} = \frac{X}{n}$, the MLE of θ. $\tilde{\theta}$ is unbiased. $\text{Var}(\tilde{\theta}) = \frac{1}{n^2}\text{Var}(X) = \theta(1-\theta)/n$. $\text{MSE}(\tilde{\theta}) = \theta(1-\theta)/n$.)

Solution

1.

$$E[\hat{\theta}] = E\left[\frac{X+1}{n+2}\right] = \frac{E[X]+1}{n+2} = \frac{n\theta+1}{n+2} \neq \theta$$

2.

$$\text{Yes,} \qquad \hat{\theta} = \frac{X/n + 1/n}{1+2/n} \overset{P}{\rightarrow} \theta \text{ as } n \rightarrow \infty.$$

$$\text{Because} \qquad X/n \overset{P}{\rightarrow} \theta \qquad \text{(SLLN)}$$

3.

$$\begin{aligned} b(\hat{\theta}) &= E[\hat{\theta}] - \theta = \frac{n\theta+1}{n+2} - \frac{n\theta+2\theta}{n+2} = \frac{1-2\theta}{n+2} \\ \text{Var}(\hat{\theta}) &= \frac{\text{Var}(X)}{(n+2)^2} = \frac{n\theta(1-\theta)}{(n+2)^2} \\ \text{MSE}(\hat{\theta}) &= \text{Var}(\hat{\theta}) + b^2(\hat{\theta}) = \frac{n\theta(1-\theta)}{(n+2)^2} + \frac{(1-2\theta)^2}{(n+2)^2} \\ &= \frac{n\theta - n\theta^2 + 4\theta^2 - 4\theta + 1}{(n+2)^2} \end{aligned}$$

A quick simulation shows that for $\theta \in (0,1)$ (exclusive), $\text{MSE}(\hat{\theta}) < \text{MSE } (\tilde{\theta})$ for n sufficiently large. (The two MSEs are asymptotically equivalent.)