Math 159, Spring 2005
Professor Hardin
Variance of the Rank-Sum w/ties

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\begin{aligned}
Var(T) &= Var\Big(\sum_{i=1}^{n} R_i\Big) = \sum_{i=1}^{n} Var(R_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} Cov(R_i, R_j) \\
Cov(R_i, R_j) &= E[R_i - \mu_{R_i}][R_j - \mu_{R_j}] \\
&= \sum_{k=1}^{N}\sum_{l=1, k\neq l}^{N} (R_k - \mu_{R_k})(R_l - \mu_{R_l}) \cdot \frac{1}{N(N-1)} \\
&= \frac{1}{N(N-1)}\Big\{\Big[\sum_{k=1}^{N}(R_k - \mu_{R_k})\Big]^2 - \sum_{k=1}^{N}(R_k - \mu_{R_k})^2\Big\} \\
&= \frac{-1}{N(N-1)}\sum_{k=1}^{N}(R_k - \mu_{R_k})^2 = \frac{-N}{N(N-1)}\sum_{k=1}^{N}(R_k - \mu_{R_k})^2\frac{1}{N} \\
&= \frac{-1}{(N-1)} Var(R_k)
\end{aligned}
$$

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\begin{aligned}
Var(T) &= \sum_{i=1}^{n} Var(R_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} Cov(R_i, R_j) \\
&= n\ Var(R_i) - \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{-1}{(N-1)} Var(R_i) \\
&= n\ Var(R_i) - \frac{n(n-1)}{(N-1)} Var(R_i) \\
&= \frac{n(N-1-n+1)}{(N-1)} Var(R_i) = \frac{nm}{N-1} Var(R_i) \\
&= \frac{nm}{N-1}\sum_{i=1}^{N}\Big(R_i - \frac{N+1}{2}\Big)^2\frac{1}{N} \\
&= \frac{nm}{N(N-1)}\sum_{i=1}^{N}(R_i^2 - 2R_i\frac{N+1}{2} + \frac{(N+1)^2}{4}) \\
&= \frac{nm}{N(N-1)}\Big[\sum_{i=1}^{N} R_i^2 - (N+1)\sum_{i=1}^{N} R_i + N\frac{(N+1)^2}{4}\Big] \\
&= \frac{nm}{N(N-1)}\Big[\sum_{i=1}^{N} R_i^2 - (N+1)\frac{N(N+1)}{2} + \frac{N(N+1)^2}{4}\Big] \\
&= \frac{nm}{N(N-1)}\Big[\sum_{i=1}^{N} R_i^2 - \frac{2N(N+1)^2}{4} + \frac{N(N+1)^2}{4}\Big] \\
&= \frac{nm}{N(N-1)}\Big[\sum_{i=1}^{N} R_i^2 - \frac{N(N+1)^2}{4}\Big]
\end{aligned}
$$