Assignment #12

Due on Monday, October 29, 2007

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.3 on *Directional Derivatives,* pp. 181–187, in Bressoud.

Background and Definitions

C^2 **Maps.** Let U denote an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a scalar field. Suppose the partial derivatives

$$\frac{\partial f}{\partial x_j}(x) \quad j = 1, 2, \ldots, n,$$

exist for all $x \in U$. If the partial derivatives of these functions exit in U, we call them the *second partial derivatives of* f in U and denote them by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad j = 1, 2, \ldots, n; i = 1, 2, \ldots, n.$$

If the second partial derivatives of f exist and are continuous in U, then we say that f *is of class* C^2, or *a* C^2 *map.*

The Divergence. Let U denote an open region in $\mathbb{R}^3$ and $F\colon U \to \mathbb{R}^3$ be a vector field given by

$$F(x, y, z) = P(x, y, z)\,\widehat{i} + Q(x, y, z)\,\widehat{j} + R(x, y, z)\,\widehat{i},$$

where P, Q and R are differentiable scalar fields in U. The *divergence* of F, denoted div F, is defined to be the scalar field

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Do the following problems

1. Let $r = \sqrt{x^2 + y^2 + z^2}$ for $(x, y, z) \in \mathbb{R}^3$. Compute the second partial derivatives of r, and give their domains of definition.

2. Let U denote an open region in $\mathbb{R}^3$ and $f\colon U \to \mathbb{R}$ be a C^2 scalar field. Compute div ∇f in terms of the second partial derivatives of f.

3. Let U denote an open region in $\mathbb{R}^3$ which does not contain the origin $(0,0,0)$ and define $f\colon U \to \mathbb{R}$ by

$$f(x,y,z) = \frac{1}{r} \quad \text{where} \quad r = \sqrt{x^2+y^2+z^2} \text{ and } \quad (x,y,z) \in U.$$

Compute $\operatorname{div} \nabla f$.

4. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ denote a scalar field whose second partial derivatives exist in D. Fix $(x,y) \in D$, and define the scalar map

$$S(h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y),$$

where $|h|$ and $|k|$ are sufficiently small.

(a) Apply the Mean Value Theorem to obtain an $\overline{x}$ in the interval $(x, x+h)$, or $(x+h, x)$ (depending on whether h is positive or negative, respectively) such that

$$S(h,k) = \left(\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y)\right) h.$$

(b) Apply the Mean Value Theorem to obtain a $\overline{y}$ in the interval $(y, y+k)$, or $(y+k, y)$ (depending on whether k is positive or negative, respectively) such that

$$S(h,k) = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y}) hk.$$

5. *(Continuation of Problem 4.)*

(c) Show that if f is of class C^2, then

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

(d) Deduce that if f is of class C^2, then

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y);$$

that is, the *mixed* second partial derivatives are the same for C^2 maps.