Assignment #13

Due on Wednesday, October 31, 2007

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.6 on *The Chain Rule,* pp. 201–205, in Bressoud.

Do the following problems

1. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ be a scalar field for which the second partial derivatives exist for all $x \in D$.

   (a) Compute the Jacobian matrix of the gradient map $\nabla f\colon \mathbb{R}^2 \to \mathbb{R}^2$.

   (b) Recall that the scalar field f is said to be of class C^2 if its second partial derivatives exist and are continuous on D.
   Prove that if f is a C^2 map, then the Jacobian matrix of ∇f is a symmetric matrix.

2. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ be a C^2 scalar field on D. The Jacobian of the gradient map $\nabla f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called the *Hessian* of the function f and is denoted by H_f; that is

$$H_f(x, y) = J_{\nabla f}(x, y).$$

   Compute the Hessian for the following scalar fields in $\mathbb{R}^2$.

   (a) $f(x, y) = x^2 - y^2$ for all $(x, y) \in \mathbb{R}^2$.

   (b) $f(x, y) = xy$ for all $(x, y) \in \mathbb{R}^2$.

3. Let A denote a symmetric $n \times n$ matrix; recall that this means that $A^T = A$, where A^T denotes the transpose of A. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \dfrac{1}{2}(Ax)\cdot x$ for all $x \in \mathbb{R}^n$; that is, $f(x)$ is the dot–product of Ax and x. In terms of matrix product,

$$f(x) = \frac{1}{2}(Ax)^T x \quad \text{for all} \quad x \in \mathbb{R}^n,$$

   where x is expressed as a column vector.

   (a) Show that f is differentiable and compute the gradient map ∇f.

   (b) Show that the gradient map ∇f is differentiable, and compute its derivative.

4. Let I be an open interval of real numbers and U ba an open subset of $\mathbb{R}^n$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path and that $f\colon U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of I under σ, $\sigma(I)$, is contained in U. Suppose also that the derivative of the path σ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I.$$

Show that if the gradient of f along the path σ is never zero, then f decreases along the path as t increases.

Suggestion: Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

5. Exercises 2 and 4 on page 207 in the text.