**Assignment #1**

**Due on Wednesday September 10, 2007**

**Read** Section 1.1 on *The Malthusian Model,* pp. 2–5, and Section 1.2 on *Nonlinear Models,* pp. 11–17, in Allman and Rhodes.

**Do** the following problems

1. Show that the solution to the difference equation

$$X_{t+1} = X_t$$

must be constant.

2. *Modeling Red Blood Cell Production*[1]. In the circulatory system, red blood cells (RBCs) are constantly being filtered out and destroyed by specialized "clean–up" cells in the spleen and liver, and replenished by the bone marrow. Since the cells carry oxygen throughout the body, their numbers must be maintained at some constant level. In problems 2–5, we model the removing of RBCs by the spleen and liver, and their replenishing by the bone marrow in order to understand how the RBC levels may be maintained.

Assume that the spleen and liver remove a fraction $f$ of the RBCs each day, and that the bone marrow produces new cells at a daily rate proportional to the number of RBCs lost on the previous day with proportionality constant $\gamma$.

Derive a system of two difference equations for $R_t$, the RBC count in circulation on day $t$, and $M_t$, the number of RBCs produced by the bone marrow on day $t$, where $t = 1, 2, 3, \ldots$

*Suggestion:* Consider the number of RBCs in circulation on day $t+1$, $R_{t+1}$. By the *conservation principle,* the change $R_{t+1} - R_t$ in the number of RBCs from day $t$ to day $t+1$ must be equal the number of new RBCs produced on day $t$ minus the number of RBCs that were removed on that same day. On the other hand, the number of new RBCs produced by the bone marrow on day $t+1$, $M_{t+1}$, must be given by the expression

$$M_{t+1} = \gamma \times (\text{Number of RBCs removed on day } t).$$

3. *Red Blood Cell Production (continued).* By considering the number of RBCs in circulation on day $t+2$, we are able to combine the two difference equations derived in the previous problem into a single difference equation of the form

$$R_{t+2} = bR_{t+1} + cR_t, \tag{1}$$

[1] Edelstein–Keshet, *Mathematical Models in Biology,* pg. 27

where $b$ and $c$ are constants. Determine expressions for $b$ and $c$ in terms of $f$ and $\gamma$.

Equation (1) is an example of a *linear second order difference equation.*

4. *Red Blood Cell Production (continued).* We may seek to find a solution to the linear second order difference equation (1) as follows:

   (a) Assume that the sought after solution is of the form $R_t = A\lambda^t$, where $A$ is some constant that will depend on the initial conditions, and $\lambda$ is a parameter that is to be determined by substituting into the difference equation. Substitute this assumed form for $R_t$ into equation (1) to obtain an expression for $\lambda$. Assuming that neither $A$ nor $\lambda$ are zero, simplify the expression to get the second order equation

   $$\lambda^2 = b\lambda + c. \tag{2}$$

   (b) Solve equation (2) for $\lambda$ to obtain two possible solutions $\lambda_1$ and $\lambda_2$ in terms of $f$ and $\gamma$, where $\lambda_1 < \lambda_2$.

   (c) Verify that $A_1\lambda_1^t$ and $A_2\lambda_2^t$, where $A_1$ and $A_2$ are arbitrary constants, both solve the difference equation (1).

   (d) Verify that

   $$R_t = A_1\lambda_1^t + A_2\lambda_2^t, \tag{3}$$

   where $A_1$ and $A_2$ are arbitrary constants, also solves the difference equation (1).

   The function $R_t$ in equation (3) is called the *general solution* of the difference equation (1).

5. *Red Blood Cell Production (continued).* Assume that 1% of the RBCs are filtered out of circulation by the spleen and liver in a day; that is $f = 0.01$.

   (a) If $\gamma = 1.50$, what does the general solution (3) predict about the RBC count as $t \to \infty$?

   (b) Suppose now that $\gamma = 0.50$. What does the general solution (3) predict about the RBC count as $t \to \infty$?

   (c) Suppose now that $\gamma = 1$. What does the general solution (3) predict about the RBC count as $t \to \infty$?

   (d) Which of the three values of $\gamma$ discussed in the previous three parts seems to yield a reasonable prediction? What implication does that have about RBC levels in the long run?