Solutions to some Review Problems for Exam 1

6. Investigate the following discrete model for a population of size N_t that is being harvested at a constant rate of H individuals per unit time:

$$\Delta N = rN\left(1 - \frac{N}{K}\right) - H,$$

where r, K and H are positive parameters.

Solution: First, find equilibrium solutions by solving the equation

$$rN\left(1 - \frac{N}{K}\right) - H = 0.$$

This yields two distinct solutions for the case $H < \dfrac{rK}{4}$; namely

$$N_1^* = \frac{K - \sqrt{K^2 - 4KH/r}}{2} \quad \text{and} \quad N_2^* = \frac{K + \sqrt{K^2 - 4KH/r}}{2},$$

or

$$N_1^* = \frac{K}{2} - \sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}} \quad \text{and} \quad N_2^* = \frac{K}{2} + \sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}}.$$

Observe that

$$0 < N_1^* < \frac{K}{2} < N_2^* < K.$$

Denote the right hand side of the equation by $g(N)$; then,

$$g(N) = rN\left(1 - \frac{N}{K}\right) - H.$$

Re–write the equation in the form

$$N_{t+1} = N_t + g(N_t),$$

and put

$$f(x) = x + g(x)$$

for all real values of x, so that

$$f'(x) = 1 + g'(x).$$

In order to apply the principle of linearized stability at N_1^*, observe that

$$g'(x) = r\left(1 - \frac{2}{K}x\right),$$

so that

$$\begin{aligned}
g'(N_1^*) &= r\left(1 - \frac{2}{K}N_1^*\right) \\
&= r\left(1 - \frac{2}{K}\left(\frac{K}{2} - \sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}}\right)\right) \\
&= \frac{2r}{K}\sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}},
\end{aligned}$$

which is positive. It then follows that $f'(N_1^*) = 1 + g'(N_1^*) > 1$, and therefore, by the principle of linearized stability, N_1^* is unstable in the case $H < \dfrac{rK}{4}$.

Doing the same calculation for N_2^* we get

$$\begin{aligned}
g'(N_2^*) &= r\left(1 - \frac{2}{K}N_2^*\right) \\
&= r\left(1 - \frac{2}{K}\left(\frac{K}{2} + \sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}}\right)\right) \\
&= -\frac{2r}{K}\sqrt{\left(\frac{K}{2}\right)^2 - \frac{KH}{r}} \\
&= -\sqrt{r^2 - \frac{4rH}{K}}.
\end{aligned}$$

For stability of N_2^* we then want that

$$\left|1 - \sqrt{r^2 - \frac{4rH}{K}}\right| < 1,$$

or

$$-1 < 1 - \sqrt{r^2 - \frac{4rH}{K}} < 1,$$

which yields

$$\sqrt{r^2 - \frac{4rH}{K}} < 2.$$

Solving the inequality for r we then get that if

$$0 < r < 2\sqrt{1 + \frac{H^2}{K^2}},$$

then N_2^* is stable.

Similarly, if $r > 2\sqrt{1 + \frac{H^2}{K^2}}$, then N_2^* is unstable.

Next, consider the case $H = \frac{rK}{4}$.

In this case the equation has only one critical point: $N^* = \frac{K}{2}$. Observe that $f'(N^*) = 1$, since $g'(N^*) = 0$, and so the principle of linearized stability does not yield any conclusion. However, observe that for any $N \neq \frac{K}{2}$, $g(N) < 0$ and so $\Delta N < 0$ and so N_t is always decreasing around $K/2$ and so this equilibrium point is also unstable. In fact if the population is below $K/2$ at some time, then it will decrease to extinction in finite time.

Finally, if $H > \frac{rK}{4}$, then there are no (real) equilibrium points. In this case, $g(N) < 0$ for all values of N and so $\Delta N < 0$ at all times. Thus, N_t is always decreasing and the population will decrease to extinction in finite time. $\square$

7. Suppose the growth of a population of size N_t at time t is dictated by the discrete model

$$N_{t+1} = \frac{400 N_t}{(10 + N_t)^2}.$$

(a) Find the biologically reasonable fixed points for this difference equation.

Solution: Let $f(x) = \frac{400x}{(10+x)^2}$ for $x \in \mathbb{R}$. Then, the fixed points of the equation are solutions of

$$f(x) = x,$$

or

$$\frac{400x}{(10+x)^2} = x.$$

To solve this equation, first write

$$\frac{400x}{(10+x)^2} - x = 0,$$

or

$$x\left(\frac{400}{(10+x)^2}-1\right)=0,$$

from which we get that

$$x=0 \quad \text{or} \quad (10+x)^2=400.$$

We therefore get that

$$x=0, \text{ or } x=-30, \text{ or } \ x=10.$$

Out of these fixed points, only the first and the last are biologically reasonable. Hence, $N_1^*=0$ and $N_2^*=10$. □

(b) Determine the stability properties of the equilibrium points found in the previous part.

Solution: Compute the derivative of f to get

$$f'(x)=\frac{400(10-x)}{(10+x)^3}.$$

Then

$$f'(N_1^*)=\frac{400(10)}{(10)^3}=4>1,$$

and therefore $N_1^*=0$ is unstable.
On the other hand, since

$$|f'(N_2^*)|=0<1,$$

$N_2^*=10$ is stable. □

(c) If $N_0=5$, what happens to the population in the long run?

Answer: $\lim\limits_{t\to\infty} N_t=10.$ □

8. *(Problem 1.1.13 on page 8 in Allman and Rhodes)*

As limnologists and oceanographers are well aware, the amount of sunlight that penetrates the various depths of water can greatly affect the communicates that live there. Assuming the water has uniform *turbidity,* the amount of light that penetrates through a 1–meter column of water is proportional to the amount entering the column.

(a) Explain why this leads to a model of the form $L_{d+1}=kL_d$, where L_d denotes the amount of light that penetrates a depth of d meters.

Solution: The expression "the amount of light that penetrates through a one–meter column of water is proportional to amount entering the column" translates into

$$L_{d+1} = kL_d,$$

where k is a constant of proportionality. □

(b) In what range must k be for this model to be physically meaningful?

Solution: The amount of light that penetrates the column of water cannot exceed the amount that comes in. We therefore require that $0 \leqslant k \leqslant 1$. □

(c) For $k = 0.25$, $L_o = 1$, plot L_d for $d = 0, 1, \ldots, 10$.

(d) Would a similar model apply to light filtering through the canopy of a forest? Is the "uniform turbidity" assumption likely to apply there?

Answer: The uniform turbidity assumption would probably not apply to this situation. □

9. *(Problem 1.2.11 on pages 18–20 in Allman and Rhodes)*

Some of the same modeling ideas and used in population studies appear in very different scientific settings.

(a) Often, chemical reactions occur at rates proportional to the amounts of raw materials present. Suppose we use a very small time interval to model such a reaction with a difference equation. Assume a fixed total amount of chemicals K, and that chemical 1, which initially occurs in amount K, is converted to chemical 2, which occurs in amount N_t at time t. Explain why

$$\Delta N = r(K - N).$$

What values of r are reasonable? What is N_o? What does the graph of N_t as a function of t look like?

Solution: If N_t is the amount of chemical 2 present at time t, the "rate of change of N being proportional to the amount of chemical 1" translates into

$$\Delta N = r(K - N)$$

where $0 < r < 1$, since the largest change in the amount of the product of the reaction cannot exceed the original amount K of reactant present.

If $N_o = 0$, the the graph of N_t begins at 0 increasing rapidly, then the rate of increase will decrease as more chemical 2 is produced. The graph of N_t will then be concave down and will level off towards K. □

(b) Chemical reactions are said to be *autocatalytic* if the rate at which they occur is proportional to both the amount of raw materials and the amount of the product (i.e., the product of the reaction is a catalyst to the reaction). We can again use a very small time interval to model such a reaction with a difference equation. Assume a fixed total amount of chemicals K, and that chemical 1 is converted to chemical 2, which occurs in amount N_t at time t. Explain why

$$\Delta N = rN(K - N).$$

If N_o is small (but not 0), What does the graph of N_t as a function of t look like? Can you explain the shape of the graph intuitively? (Note that r will be very small because we are using a small time interval.) The logistic growth model is sometimes also referred ao to as the *autocatalytic* model.

***Solution**:* Let N be as in the previous part. Then, the rate of change of of N being proportional to product of the amounts of both chemicals translates into

$$\Delta N = rN(K - N)$$

This is the logistic equation. For $N_o \neq 0$ but close to zero, the graph of N_t will have the logistic, or S–shape form leveling off at K. If $N_o = 0$, then $N_t = 0$ for all t since 0 is an equilibrium point of the equation. The S-shape from of the graph can be explained by noting that for small amounts of N, the rate of growth is small; this rate of growth will increase to a largest value at $K/2$, and after that will begin to decrease again. The graph will then level off at K. □