Review Problems for Exam 2

1. A patient is given the drug theophylline intravenously at a constant rate of 43.2 mg/hour to relieve acute asthma. You can imagine the drug as entering a compartment of volume $35,000$ ml. (This is an estimate of the volume of the part of the body through which the drug circulates.) The rate at which the drug leaves the patient is proportional to the quantity there, with proportionality constant 0.082.

   (a) Write a differential equation for the quantity, $Q = Q(t)$, in milligrams, of the drug in the body at time t hours.

   (b) Give the equilibrium solution, $\overline{Q}$, to the equation in part (a).

   (c) Assuming that the patient's body contains none of the drug initially, give $Q(t)$ for all t, and sketch an approximate graph of Q as a function of t.

   (d) What is the limiting value of $Q(t)$ as $t \to \infty$?

2. *[Harvesting]* The following differential equation models the growth of a population of size $N = N(t)$ that is being harvested at a rate proportional to the population density

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{1}$$

   where r, K and E are parameters and non–negative parameters with $r > 0$ and $K > 0$.

   (a) Give an interpretation for this model. In particular, give interpretation for the term EN. The parameter E is usually called the harvesting *effort.*

   (b) Calculate the equilibrium points for the equation (1), and give conditions on the parameters that yield a biologically meaningful equilibrium point. Determine the nature of the stability of that equilibrium point. Sketch possible solutions to the equation in this situation.

   (c) What does the model predict if $E \geq r$?

3. *[Harvesting, continued]* Suppose that $0 < E < r$ in equation (1), and let $\overline{N}$ denote the positive equilibrium point. The quantity $Y = E\overline{N}$ is called the *harvesting yield.*

   (a) Find the value of E for which the harvesting yield is the largest possible; this value of the yield is called the *maximum sustainable yield.*

   (b) What is the value of the equilibrium point for which there is the maximum sustainable yield?

4. We have seen that the (continuous) logistic model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

   where r and K are positive parameters, has an equilibrium point at $\overline{N} = K$.

(a) Let $g(N) = rN\left(1 - \frac{N}{K}\right)$ and give the linear approximation to $g(N)$ for N close to K:

$$g(K) + g'(K)(N - K).$$

Observe that $g(K) = 0$ since K is an equilibrium point.

(b) Let $u = N - K$ and consider the linear differential equation

$$\frac{du}{dt} = g'(K)u.$$

This is called the *linearization* of the equation

$$\frac{dN}{dt} = g(N)$$

around the equilibrium point $\overline{N} = K$. Use separation of variables to solve this equation. What happens to $|u(t)|$ as $t \to \infty$, where u is any solution to the linearized equation?

(c) Use your result in the previous part to give an explanation as to why any solution to the logistic equation that begins very close to K can be approximation by

$$K + u(t),$$

where u is a solution to the linearized equation.

(d) Suppose that $N = N(t)$ is a solution to the logistic equation that starts at N_o, where N_o is very close to K. Find an estimate of the time it takes for the distance $|N(t) - K|$ to decrease by a factor of e. This time is called the *recovery time.*

5. Imagine a culture grown from a single bacterium. Suppose that there have been n division cycles. Assume that no bacterium has died during those cycles.

   (a) How large is the culture? How many divisions have there been? Assume that all divisions that occur during the same cycle happen at the same time (these are usually referred to as *synchronous divisions*).

   (b) Recall that the mutation rate, a, gives the probability that a given bacterium will mutate during a division. Let N denote the total bacterial population in a culture grown out of a single bacterium in n division cycles. Show that the probability, p_o, of no mutants present after the n division cycles can be approximated by $e^{-\mu}$, where $\mu = aN$ and N is very large.

   Suggestion: If D is the number of divisions that have occurred in n division cycles, what is the probability that no mutation has occurred in any of those divisions? What happens to this probability as N tends to infinity?

(c) There will be exactly one mutant in the culture after n division cycles if no mutation occurs in the first $n-2$ cycles, and exactly one mutation occurs in the $(n-1)^{\text{st}}$ cycle.

i. Explain why the probability of one mutation in the $(n-1)^{\text{st}}$ cycle is $a \cdot 2^{n-1}$.

ii. Estimate the probability, p_1, that there will be exactly one mutant in the culture after n division cycles, if the culture size, N, is very large.
Suggestion: If D is the number of divisions that have occurred in n division cycles, what is the probability that no mutation has occurred in $D-1$ of those divisions, and exactly one mutation occurs in one division? What happens to this probability as N tends to infinity?

(d) If the number of mutants, r, in the culture is equal to 2, two bacteria might have mutated during the $n-1$ division cycle, or one bacterium might have mutated during the $n-2$ cycle giving rise to 2 mutants after cell division in the $n-1$ cycle. Estimate the probability, p_2, of this event for N very large.

(e) Use your results in the previous three parts to estimate the probability that there will be 3 or more resistant bacteria in the culture after n division cycles when the population size, N, is very large.

6. Suppose we are interested in tracking the proportions of the genotypes GG, Gg and gg in a very large population, and that, initially, those proportions are p_o, q_o and r_o, respectively.

(a) Let a denote the proportion of allele G in the entire population and b be that of allele g. Find a and b in terms of p_o, q_o and r_o.

(b) Assuming random mating between individuals of the various genotypes, compute the proportions p_1, q_1 and r_1 of the genotypes GG, Gg and gg, respectively, in the first generation.

(c) Verify that p_1, q_1 and r_1 are given by a^2, $2ab$ and b^2, respectively, where a and b are as in part (a).