Assignment #10

Due on Friday, October 10, 2008

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.3 on *Directional Derivatives,* pp. 181–187, in Bressoud.

Background and Definitions

Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset U of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0} \frac{f(x+t\widehat{u}) - f(x)}{t}$$

exists, we call it the *directional derivative of f at x in the direction of the unit vector $\widehat{u}$.* We denote it by $D_{\widehat{u}}f(x)$.

If f is differentiable at $x \in U$, then

$$D_{\widehat{u}}f(x) = \nabla f(x) \cdot \widehat{u},$$

where $\nabla f(x)$ is the gradient of f at x.

Do the following problems

1. Let v denote a vector in $\mathbb{R}^n$ and suppose that $v \cdot \widehat{u} = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$. Prove that v must be the zero vector.

   Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $x \in U$. Prove that if $D_{\widehat{u}}f(x) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(x)$ must be the zero vector.

2. The scalar field $f\colon U \to \mathbb{R}$ is said to have a *local minimum* at $x \in U$ if there exists $r > 0$ such that $B_r(x) \subseteq U$ and

   $$f(x) \leqslant f(y) \quad \text{for every} \quad y \in B_r(x).$$

   Prove that if f is differentiable at $x \in U$ and f has a local minimum at x, then $\nabla f(x) = \mathbf{0}$, the zero vector in $\mathbb{R}^n$.

   (*Suggestion:* Note that for $|t| < r$ and any unit vector $\widehat{u}$ in $\mathbb{R}^n$,

   $$f(x) \leqslant f(x + t\widehat{u}).$$

   It then follows that

   $$f(x+t\widehat{u}) - f(x) \geqslant 0.$$

   Divide by $t \neq 0$ and then let $t \to 0$. Consider the two cases $t > 0$ and $t < 0$ separately.)

3. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $x \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}}f(x)$ is $\|\nabla f(x)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(x)$.

4. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for } \ 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function g is well defined.

   (b) Show that g is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for } \ 0 < t < 1.$$

   (*Suggestion:* Consider

$$\frac{g(t+h) - g(t)}{h} = \frac{f(x + t(y-x) + h(y-x)) - f(x + t(y-x))}{h}$$

   and apply the definition of differentiability of f at the point $x + t(y - x)$.)

   (c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}}f(z)\|y - x\|,$$

   where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$.
   (*Hint:* Observe that $g(1) - g(0) = f(y) - f(x)$.)

   (d) Deduce that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(x) = \mathbf{0}$ for all $x \in U$, then f must be a constant function.

5. Exercise 13 on page 198 in the text.