Assignment #13

Due on Wednesday, October 29, 2008

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.6 on *The Chain Rule,* pp. 201–205, in Bressoud.

Do the following problems

1. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ be a C^2 scalar field on D. The Jacobian of the gradient map $\nabla f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called the *Hessian* of the function f and is denoted by H_f; that is

$$H_f(x, y) = J_{\nabla f}(x, y).$$

   Compute the Hessian for the following scalar fields in $\mathbb{R}^2$.

   (a) $f(x, y) = x^2 - y^2$ for all $(x, y) \in \mathbb{R}^2$.

   (b) $f(x, y) = xy$ for all $(x, y) \in \mathbb{R}^2$.

2. Let A denote a symmetric $n \times n$ matrix; recall that this means that $A^T = A$, where A^T denotes the transpose of A. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \dfrac{1}{2}(Ax)\cdot x$ for all $x \in \mathbb{R}^n$; that is, $f(x)$ is the dot–product of Ax and x. In terms of matrix product,

$$f(x) = \frac{1}{2}(Ax)^T x \quad \text{for all} \quad x \in \mathbb{R}^n,$$

   where x is expressed as a column vector.

   (a) Show that f is differentiable and compute the gradient map ∇f.

   (b) Show that the gradient map ∇f is differentiable, and compute its derivative.

3. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

   Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

4. Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c,d] \to [a,b]$ be a one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c,d]$. Define

$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c,d].$$

$\gamma\colon [c,d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ

(a) Show that γ is a differentiable, one–to–one path.

(b) Compute $\gamma'(t)$ and show that it is never the zero vector.

(c) Show that σ and γ have the same image in $\mathbb{R}^n$.

5. Exercise 8 on page 208 in the text.